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Construction of multivector inverse for Clifford algebras over $2m+1$-dimensional vector spaces from multivector inverse for Clifford algebras over $2m$-dimensional vector spaces.*

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Abstract. Assuming known algebraic expressions for multivector inverses in any Clifford algebra over an even dimensional vector space $\mathbb{R}^{p',q'}$, $n' = p' + q' = 2m$, we derive a closed algebraic expression for the multivector inverse over vector spaces one dimension higher, namely over $\mathbb{R}^{p,q}$, $n = p+q = p'+q'+1 = 2m+1$. Explicit examples are provided for dimensions $n' = 2, 4, 6$, and the resulting inverses for $n = n' + 1 = 3, 5, 7$. The general result for $n = 7$ appears to be the first ever reported closed algebraic expression for a multivector inverse in Clifford algebras $Cl(p, q)$, $n = p + q = 7$, only involving a *single* addition of multivector products in forming the determinant.

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1. Introduction

The inverse of Clifford algebra multivectors is useful for the instant solution of multivector equations, like $AXB = C$, which gives with the inverses of A and B the solution $X = A^{-1}CB^{-1}$, $A, B, C, X \in Cl(p, q)$. Furthermore the inverse of geometric transformation versors is essential for computing two-sided spinor- and versor transformations as $X' = V^{-1}XV$, where V can in principle be factorized into a product of vectors from $\mathbb{R}^{p,q}$, $X \in Cl(p, q)$.

* This paper is dedicated to the Turkish journalist Dennis Yücel, who is since 14. Feb. 2017 a political prisoner in the Silivri prison, west of Istanbul, without having been formally charged before a legal court even at the time of writing this paper (Jan. 2018). His immediate and unconditional release is being hoped for. [7]

Previous work on the multivector inverse include [3, 12, 6] for dimensions $n = p + q < 6$, and recently [2] for dimension $n = 6$. For vector space dimensions $n = 1$ to 5, [6] exclusively employed compact signature independent abstract algebra computations. Multivector determinants[1] for $n = 1$ to 5 were also studied in [9, 10] (calling them norm functions). For $n = 6$, [2] employed extensive symbolic computer algebra computations. The current paper is the first to report: (1) A general method to construct the multivector inverse for any Clifford algebra with $n = p+q = 2m+1$ from the known multivector inverse of Clifford algebras over vector spaces of one dimension less, i.e. from the multivector inverse in Clifford algebras with $n' = p+q-1 = 2m$. (2) Applying this new method, we show for the first time how to directly construct the multivector inverse for Clifford algebra multivectors in any $Cl(p, q)$, $n = p + q = 7$, that can be applied without intermediate use of matrix isomorphisms, and involves only a *single* addition of multivector products in forming the determinant. The latter fact promises to increase the speed and accuracy of numerical computations of multivector inverses for $n = 7$, as well as the speed of symbolic multivector inverse computations.

The paper is structured as follows. Section 2 reviews certain properties of Clifford algebras, needed in the remainder of the paper. Section 3 shows explicit examples for constructing compact multivector inverse expressions in Clifford algebras $Cl(p, q)$, $n = p+q = 3, 5, 7$, based on known algebraic multivector inverse expressions in Clifford algebras $Cl(p', q')$, $n' = p' + q' = 2, 4, 6$. Section 4 explains the general method for constructing compact multivector inverse expressions in Clifford algebras $Cl(p, q)$, $n = p + q = 2m + 1$, assuming known algebraic multivector inverse expressions in Clifford algebras $Cl(p', q')$, $n' = p' + q' = 2m$ for any value of $m \in \mathbb{N}$. Finally, Section 5 concludes the paper, followed by acknowledgments and references.

2. Preliminaries

For an introduction to Clifford algebra we refer to the popular textbook [8], or to a tutorial, like [4]. A software package, which we used for verifying all our examples, that combines multivector computations and matrix computations, is the Clifford Multivector Toolbox [11] for MATLAB. In the following we state some important facts about complex numbers $\mathbb{C}$, hyperbolic numbers (split complex numbers), and Clifford algebras $Cl(p, q)$ over quadratic vector spaces $\mathbb{R}^{p,q}$.

For complex numbers $a+ib \in \mathbb{C}$, $a, b \in \mathbb{R}$, we define complex conjugation as

$$\mathrm{cc}(a + ib) = a - ib, \tag{1}$$

which allows to compute the square of the modulus of a complex number as

$$|a + ib|^2 = (a + ib)\mathrm{cc}(a + ib) = (a + ib)(a - ib) = a^2 + b^2. \tag{2}$$

[1] We thank an anonymous reviewer for this interesting reference.

For hyperbolic (split complex) numbers, with hyperbolic unit u, $u^2 = 1$, we define hyperbolic conjugation as

$$\mathrm{hc}(a + ub) = a - ub, \tag{3}$$

which allows to compute the square of the modulus of a hyperbolic number as

$$|a + ub|^2 = (a + ub)\mathrm{hc}(a + ub) = (a + ub)(a - ub) = a^2 - b^2. \tag{4}$$

Complex numbers can be represented by 2×2 real matrices as

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{5}$$

The real determinant of this matrix is identical to the square of the modulus

$$\det \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a^2 + b^2. \tag{6}$$

Similarly, hyperbolic numbers can be represented by 2×2 real matrices as

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}. \tag{7}$$

The real determinant of this matrix is also identical to the square of the modulus

$$\det \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a^2 - b^2. \tag{8}$$

In this paper we employ reversion $\widetilde{A}$, grade involution $\mathrm{gi}(A) = \widehat{A}$, Clifford conjugation $\overline{A} = \widehat{\widetilde{A}} = \widetilde{\widehat{A}}$, and selective grade wise involutions $\mathrm{m}_{\bar{a},\bar{b},\bar{c}}(\ldots)$ (negations of grades a, b, c, specified as indexes with overbars $\bar{a}, \bar{b}, \bar{c}$). Using the grade extraction operator $\langle \ldots \rangle_k$, $0 \leq k \leq n$, we can express these involutions as

$$\begin{aligned} \widetilde{A} &= \sum_{k=0}^{k=n} (-1)^{k(k-1)/2} \langle A \rangle_k, \\ \widehat{A} &= \sum_{k=0}^{k=n} (-1)^{k} \langle A \rangle_k, \\ \overline{A} &= \sum_{k=0}^{k=n} (-1)^{k(k+1)/2} \langle A \rangle_k, \\ \mathrm{m}_{\bar{a},\bar{b},\bar{c}}(A) &= A - 2\langle A \rangle_a - 2\langle A \rangle_b - 2\langle A \rangle_c. \end{aligned} \tag{9}$$

We apply the following Clifford algebra isomorphisms [8] for even subalgebras $Cl^+(p, q)$,

$$Cl(p, q) \cong Cl^+(p, q + 1), \qquad Cl(p, q) \cong Cl^+(q + 1, p). \tag{10}$$

Furthermore, the center Z of a Clifford algebra $Cl(p, q)$, over the odd dimensional vector space $\mathbb{R}^{p,q}$, $n = p + q = 2m + 1$, is non-trivial

$$Z = \{1, I = e_1 e_2 \ldots e_n\}. \tag{11}$$

That means every Clifford algebra $Cl(p,q)$, $n = p+q = 2m+1$, can be split into its even subalgebra and the *central* pseudoscalar I times another copy of the even subalgebra

$$Cl(p,q) = Cl^+(p,q) \oplus Cl^-(p,q) = Cl^+(p,q) \oplus Cl^+(p,q)I. \tag{12}$$

Therefore every $A \in Cl(p,q)$, $n = p+q = 2m+1$, can be represented with the help of two elements $\langle A\rangle_+, \langle B\rangle_+ \in Cl^+(p,q)$ from its even subalgebra as

$$A = \langle A\rangle_+ + \langle A\rangle_- = \langle A\rangle_+ + \langle B\rangle_+ I, \tag{13}$$

where $\langle B\rangle_+$ is the *dual* of $\langle A\rangle_-$,

$$\langle B\rangle_+ = \langle A\rangle_-^* = \langle A\rangle_- I^{-1}. \tag{14}$$

That in turn means, that every $A \in Cl(p,q)$, $n = p+q = 2m+1$, can be written as an element of the even subalgebra $Cl^+(p,q)$ with complex (for $I^2 = -1$) or hyperbolic (for $I^2 = +1$) coefficients.

For the case of $I^2 = -1$, the multivector A can be represented as a 2×2 matrix with entries from its even subalgebra

$$\begin{pmatrix} \langle A\rangle_+ & \langle B\rangle_+ \\ -\langle B\rangle_+ & \langle A\rangle_+ \end{pmatrix}. \tag{15}$$

The determinant of the above matrix is

$$\det \begin{pmatrix} \langle A\rangle_+ & \langle B\rangle_+ \\ -\langle B\rangle_+ & \langle A\rangle_+ \end{pmatrix} = \det\langle A\rangle_+ \det\langle A\rangle_+ + \det\langle B\rangle_+ \det\langle B\rangle_+. \tag{16}$$

For the case of $I^2 = +1$, the multivector A can also be represented as a 2×2 matrix with entries from its even subalgebra

$$\begin{pmatrix} \langle A\rangle_+ & \langle B\rangle_+ \\ \langle B\rangle_+ & \langle A\rangle_+ \end{pmatrix}. \tag{17}$$

The determinant of the above matrix is in this case

$$\det \begin{pmatrix} \langle A\rangle_+ & \langle B\rangle_+ \\ \langle B\rangle_+ & \langle A\rangle_+ \end{pmatrix} = \det\langle A\rangle_+ \det\langle A\rangle_+ - \det\langle B\rangle_+ \det(\langle B\rangle_+. \tag{18}$$

This shows, that knowing the determinants of the two components $\langle A\rangle_+$, $\langle B\rangle_+ \in Cl^+(p,q)$, permits easily to compute the determinant of the original full multivector $A \in Cl(p,q)$.

Note, that because every Clifford algebra $Cl(p,q)$ is isomorphic to some ring of real square matrices [8], every left inverse multivector is also right inverse.

3. Multivector inverse for Clifford algebras over $n = 3, 5, 7$-dimensional vector spaces from multivector inverse for Clifford algebras over $n' = 2, 4, 6$-dimensional vector spaces

Based on the even subalgebra based complex (or hyperbolic) representation of multivectors $\in Cl(p,q)$, $n = p+q = 2m+1$, given in (13), we now demonstrate with the help of examples for $n = 3, 5, 7$, how the known inverse of multivectors in lower dimensional Clifford algebras isomorphic to $Cl^+(p,q)$, naturally leads to closed form algebraic expressions for the multivector inverse in $Cl(p,q)$, $n = p+q = 2m+1$, itself.

3.1. Multivector inverse for Clifford algebras over 3-dimensional vector spaces from multivector inverse for Clifford algebras over 2-dimensional vector spaces

The isomorphisms (10) for Clifford algebras over 3-dimensional vector spaces are the following[2]

$$\begin{aligned} &Cl^+(3,0) \cong Cl^+(0,3) \cong Cl(0,2) \cong \mathbb{H}, \\ &Cl^+(2,1) \cong Cl^+(1,2) \cong Cl(2,0) \cong Cl(1,1). \end{aligned} \tag{19}$$

Moreover the pseudoscalar $I = e_1e_2e_3 = e_{123}$ squares to -1 for $Cl(3,0)$ and $Cl(1,2)$; while it squares to $+1$ for $Cl(2,1)$ and $Cl(0,3)$, respectively.

As a first example for the complex case, we take in the Clifford algebra $Cl(3,0)$ with $I^2 = -1$:

$$\begin{aligned} A &= 1 + 2e_1 + 3e_2 + 4e_3 + 5e_{23} + 6e_{31} + 7e_{12} + 8I \\ &= 1 + 8I + (5-2I)e_{23} + (6-3I)e_{31} + (7-4I)e_{12}. \end{aligned} \tag{20}$$

Replacing $I \to i$, we isomorphically map A to A_c in the *complexified* even subalgebra $Cl^+(3,0)$

$$A_c = 1 + 8i + (5-2i)e_{23} + (6-3i)e_{31} + (7-4i)e_{12}. \tag{21}$$

As an example for the hyperbolic case, we take in the Clifford algebra $Cl(2,1)$ with $I^2 = +1$:

$$\begin{aligned} B &= 1 + 2e_1 + 3e_2 + 4e_3 + 5e_{23} + 6e_{31} + 7e_{12} + 8I \\ &= 1 + 8I + (5+2I)e_{23} + (6+3I)e_{31} + (7-4I)e_{12}. \end{aligned} \tag{22}$$

Replacing $I \to u$, we isomorphically map B to B_h in the *hyperbolically* complexified even subalgebra $Cl^+(2,1)$

$$B_h = 1 + 8u + (5+2u)e_{23} + (6+3u)e_{31} + (7-4u)e_{12}. \tag{23}$$

Under the assumption, that in the lower dimensional Clifford algebra, isomorphic by (19) to the even subalgebra $Cl^+(p,q)$ of $Cl(p,q)$, $n = p+q = 2m+1 = 3$, the inverse of a general multivector is known, we can therefore derive an inverse for every multivector in $Cl(p,q)$, $n = p+q = 3$, as well, provided that its determinant is non-zero. We further illustrate this by way of example. By (19) the even subalgebra $Cl^+(3,0) \cong Cl(0,2)$, the quaternions. The inverse of a non-zero $a = a_0 + a_1e_{23} + a_2e_{31} + a_3e_{12} \in Cl^+(3,0)$, $a_0, a_1, a_2, a_3 \in \mathbb{R}$, can therefore be specified as

$$a^{-1} = \frac{\widetilde{a}}{a\widetilde{a}}, \tag{24}$$

[2]Note, that these isomorphisms do partially *change* the grades of basis elements. For example in $Cl^+(3,0) \cong Cl(0,2)$, two basis bivectors of $Cl^+(3,0)$ are mapped to vectors in $Cl(0,2)$, and only one basis bivector is mapped to the bivector in $Cl(0,2)$.

where we use the reverse[3] $\widetilde{a}$. Note, that the scalar $a\widetilde{a} \in \mathbb{R}$ represents the determinant of a, e.g., in the complex matrix representation of a in the matrix ring $M(2,\mathbb{C})$. Now we compute the denominator $a\widetilde{a}$ of this inverse by inserting the *full* multivector $A_c = 1 + 8i + (5-2i)e_{23} + (6-3i)e_{31} + (7-4i)e_{12}$ from the first example (21) above and obtain

$$\begin{aligned} y = A_c\widetilde{A_c} &= (1 + 8i + (5-2i)e_{23} + (6-3i)e_{31} + (7-4i)e_{12}) \\ &\quad (1 + 8i - (5-2i)e_{23} - (6-3i)e_{31} - (7-4i)e_{12}) \\ &= (1+8i)^2 + (5-2i)^2 + (6-3i)^2 + (7-4i)^2 = 18 - 96i. \end{aligned} \tag{25}$$

Please note, that on purpose, we have *not applied* the reversion operation to the complex unit i, only to the bivectors $\widetilde{e_{23}} = -e_{23}$, $\widetilde{e_{31}} = -e_{31}$, $\widetilde{e_{12}} = -e_{12}$. In this way, we have obtained a *complex* scalar (25). We apply complex conjugation[4] to this scalar to finally get the real determinant of $A \in Cl(3,0)$ of (21) (and due to the isomorphism (19) also the determinant of (20)) as

$$\begin{aligned} z = \det(A) &= y\,\mathrm{cc}(y) = A_c\widetilde{A_c}\,\mathrm{cc}(A_c\widetilde{A_c}) \\ &= (18-96i)\,(18+96i) = 9540. \end{aligned} \tag{26}$$

This allows us to establish the inverse of $A \in Cl(3,0)$ of (20) in two steps as follows

$$\begin{aligned} A_c^{-1} &= \frac{\widetilde{A_c}\,\mathrm{cc}(y)}{z} \\ &= \frac{(1 + 8i - (5-2i)e_{23} - (6-3i)e_{31} - (7-4i)e_{12})\,(18+96i)}{9540} \\ &= \frac{-750 + 240i + (-282-444i)e_{23} + (-396-522i)e_{31} + (-510-600i)e_{12}}{9540}. \end{aligned} \tag{27}$$

Finally isomorphically mapping back $i \to I$, $A_c \to A$, and multiplying out $Ie_{23} = e_{123}e_{23} = -e_1$, etc., we obtain the desired inverse of A of (20) as

$$A^{-1} = \frac{-750 + 444e_1 + 522e_2 + 600e_3 - 282e_{23} - 396e_{31} - 510e_{12} + 240e_{123}}{9540}. \tag{28}$$

We can similarly apply our method to our second example of (22) of the even subalgebra $Cl^+(2,1) \cong Cl(2,0)$, see (19). The inverse of a non-zero

[3] Note, that using the reverse, means to stay in $Cl(3,0)$, but to not apply reversion to the pseudoscalar I, respectively to i. If we would use the isomorphism to quaternions $\mathbb{H}$, then instead of reverse, quaternion conjugation would have to be used. If we would use the isomorphism to $Cl(0,2)$, then Clifford conjugation (combining reverse with grade involution) would have to be used.

[4] Note, that in (26), we could already use the back isomorphism $i \to I$, and then instead of using complex conjugation, simply use grade involution to change in the second factor $I \to \widehat{I} = -I$. This very same method could also be applied to (31). The advantage being, that grade involution can always be applied to this effect, independent of the positive or negative square of I.

$b = b_0 + b_1 e_{23} + b_2 e_{31} + b_3 e_{12} \in Cl^+(2,1)$, $b_0, b_1, b_2, b_3 \in \mathbb{R}$, can therefore be specified as

$$b^{-1} = \frac{\widetilde{b}}{b\widetilde{b}}, \tag{29}$$

again using the reverse $\widetilde{b}$, $b\widetilde{b} \in \mathbb{R}$. Now we compute the denominator $b\widetilde{b}$ of this inverse by inserting the *full* multivector $B_h = 1 + 8u + (5+2u)e_{23} + (6+3u)e_{31} + (7-4u)e_{12}$ from the second example (23) above and obtain

$$\begin{aligned} y = B_h\widetilde{B_h} &= (1 + 8u + (5+2u)e_{23} + (6+3u)e_{31} + (7-4u)e_{12}) \\ &\quad (1 + 8u - (5+2u)e_{23} - (6+3u)e_{31} - (7-4u)e_{12}) \\ &= (1+8u)^2 - (5+2u)^2 - (6+3u)^2 + (7-4u)^2 = (56-96u). \end{aligned} \tag{30}$$

Please note again, that on purpose, we have *not applied* the reversion operation to the hyperbolic unit h, only to the bivectors e_{23}, e_{31}, e_{12}. In this way, we have obtained a *hyperbolic* scalar. We apply hyperbolic conjugation to this scalar to finally get the real determinant of $B \in Cl(2,1)$ of (23) (and due to the isomorphism (19) also the determinant of (22)) as as

$$\begin{aligned} z = \det(B) = y\,\mathrm{hc}(y) &= B_h\widetilde{B_h}\,\mathrm{hc}(B_h\widetilde{B_h}) \\ &= (56-96u)(56+96u) = -6080. \end{aligned} \tag{31}$$

This allows us to establish the inverse of $B \in Cl(2,1)$ of (22) in two steps as follows

$$\begin{aligned} B_h^{-1} &= \frac{\widetilde{B}_h\,\mathrm{hc}(y)}{z} \\ &= \frac{(1 + 8u - (5+2u)e_{23} - (6+3u)e_{31} - (7-4u)e_{12})\,(56+96u)}{-6080} \\ &= \frac{(1 + 8u - (5+2u)e_{23} - (6+3u)e_{31} - (7-4u)e_{12})\,(7+12u)}{-760} \\ &= \frac{103 + 68u + (-59-74u)e_{23} + (-78-93u)e_{31} + (-1-56u)e_{12}}{-760}. \end{aligned} \tag{32}$$

Finally isomorphically mapping back $u \to I$, $B_h \to B$, and multiplying out $Ie_{23} = e_{123}e_{23} = +e_1$, etc., we obtain the desired inverse of B of (22) as

$$B^{-1} = \frac{103 - 74e_1 - 93e_2 + 56e_3 - 59e_{23} - 78e_{31} - e_{12} + 68e_{123}}{-760}. \tag{33}$$

3.2. Multivector inverse for Clifford algebras over 5-dimensional vector spaces from multivector inverse for Clifford algebras over 4-dimensional vector spaces

The isomorphisms (10) for the six Clifford algebras over $n = p + q = 5$-dimensional vector spaces are the following

$$\begin{aligned} Cl^+(5,0) &\cong Cl^+(0,5) \cong Cl(0,4), \\ Cl^+(4,1) &\cong Cl^+(1,4) \cong Cl(4,0) \cong Cl(1,3), \\ Cl^+(3,2) &\cong Cl^+(2,3) \cong Cl(3,1) \cong Cl(2,2). \end{aligned} \tag{34}$$

Moreover the central pseudoscalar[5] $I = e_1e_2e_3e_4e_5 = e_{12345}$ squares to -1 for $Cl(0,5)$, $Cl(2,3)$ and $Cl(4,1)$; while it squares to $+1$ for $Cl(5,0)$, $Cl(3,2)$ and $Cl(1,4)$, respectively.

A general multivector $Cl(p,q)$, $n = p+q = 5$, can be represented grade wise as

$$\begin{aligned} A &= \alpha + GI + B + CI + F + \beta I = \alpha + \beta I + B + CI + F + GI \\ &= \alpha + B + F + (\beta + C + G)I = \langle A\rangle_+ + (\langle A\rangle_- I^{-1})I = \langle A\rangle_+ + \langle A\rangle_-^* I, \end{aligned} \tag{35}$$

with even grade dual of the odd part $\langle A\rangle_-^* = \langle A\rangle_- I^{-1}$, scalars $\alpha, \beta \in \mathbb{R}$, bivectors $B, C \in Cl^2(p,q)$, and 4-vectors $F, G \in Cl^4(p,q)$, such that vector $GI \in \mathbb{R}^{p,q}$, and trivector $CI \in Cl^3(p,q)$. Expressed as above, $A \in Cl(p,q)$, $p+q = 5$ takes the form of a complex (or hyperbolic) scalar[6] $\alpha + \beta i$ plus a complex (or hyperbolic) bivector $B + Ci$ plus a complex (or hyperbolic) 4-vector $F + Gi$. And both $\langle A\rangle_+, \langle A\rangle_-^* \in Cl^+(p,q)$. Via the isomorphisms (34), we can isomorphically map the even part[7] $\langle A\rangle_+ = \alpha + B + F$ to a Clifford algebra $Cl(p, q-1)$ or $Cl(q, p-1)$ of a four-dimensional ($n' = p + q - 1 = 5 - 1 = 4$) vector space.

Assuming any known inverse formula for multivectors of non-zero determinant in $Cl(p, q-1)$ or $Cl(q, p-1)$, we can compute the inverse of the even grade part $\langle A\rangle_+$. The determinant of $\langle A\rangle_+$ will be real, but if we replace the coefficients by complex (respectively hyperbolic) numbers, the determinant will be complex (or hyperbolic). The product of this determinant with its complex (or hyperbolic) conjugate will produce a real number, which is identical to the determinant of the full multivector $A \in Cl(p,q)$, $p+q = 5$, and therefore leads (if non-zero) to an inverse for the full multivector $A \in Cl(p,q)$,

[5]We thank Prof. Jacques Helmstetter for pointing out the important role of the non-trivial center in Clifford algebras over $n = p + q = 5$-dimensional vector spaces.

[6]Here we apply the isomorphism to the complexified even subalgebra $Cl^+(p,q)$, where i now stands both for the complex imaginary unit (case $I^2 = -1$) and the hyperbolic unit (case $I^2 = +1$).

[7]Correspondingly, the full multivector A is mapped to the complexified (hyperbolic complexified) isomorphic algebra.

$p+q=5$. The even subalgebra $Cl^+(p,q)$ of $Cl(p,q)$, $p+q=5$, has the following even grade element basis

$$\begin{aligned}
&\{1,\\
&E_1=e_{12}, E_2=e_{13}, E_3=e_{14}, E_4=e_{15},\\
&e_{23}, e_{24}, e_{25}, e_{34}, e_{35}, e_{45},\\
&e_{1234}, e_{1235}, e_{1245}, e_{1345}, e_{2345}\}
\end{aligned} \tag{36}$$

which can be rewritten, even basis blade by even basis blade, in terms of the isomorphic lower dimensional Clifford algebra (34), generated from the basis $\{E_1, E_2, E_3, E_4\}$, as ($s=-e_1^2=\pm 1$, depending on the square of e_1)

$$\begin{aligned}
&\{1,\\
&E_1, E_2, E_3, E_4,\\
&sE_{12}, sE_{13}, sE_{14}, sE_{23}, sE_{24}, sE_{34},\\
&sE_1E_{23}=sE_{123}, sE_1E_{24}=sE_{124}, sE_1E_{34}=sE_{134}, sE_2E_{34}=sE_{234},\\
&E_{12}E_{34}=E_{1234}\}.
\end{aligned} \tag{37}$$

That means, that the even multivector $\langle A\rangle_+=\alpha+B+F$ can be represented isomorphically in two ways

$$\begin{aligned}
\langle A\rangle_+ &= \alpha 1 + B_{12}e_{12} + B_{13}e_{13} + B_{14}e_{14} + B_{15}e_{15}\\
&\quad + B_{23}e_{23} + B_{24}e_{24} + B_{25}e_{25} + B_{34}e_{34} + B_{35}e_{35} + B_{45}e_{45}\\
&\quad + F_{1234}e_{1234} + F_{1235}e_{1235} + F_{1245}e_{1245} + F_{1345}e_{1345}\\
&\quad + F_{2345}e_{2345}\\
&= \alpha 1 + B_{12}E_1 + B_{13}E_2 + B_{14}E_3 + B_{15}E_4\\
&\quad + sB_{23}E_{12} + sB_{24}E_{13} + sB_{25}E_{14} + sB_{34}E_{23} + sB_{35}E_{24} + sB_{45}E_{34}\\
&\quad + sF_{1234}E_{123} + sF_{1235}E_{124} + sF_{1245}E_{134} + sF_{1345}E_{234}\\
&\quad + F_{2345}E_{1234}.
\end{aligned} \tag{38}$$

With respect to the $16=2^4$-dimensional blade basis

$$\begin{aligned}
&\{1,\\
&E_1, E_2, E_3, E_4,\\
&E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34},\\
&E_{123}, E_{124}, E_{134}, E_{234},\\
&E_{1234}\}
\end{aligned} \tag{39}$$

of $Cl(p,q-1)$ or $Cl(q,p-1)$, $n'=p+q-1=4$, isomorphic by (34), to $Cl^+(p,q)$, we now compute from $\langle A\rangle_+$ the real scalar (see [6])

$$\langle A\rangle_+\overline{\langle A\rangle_+}m_{\bar{3},\bar{4}}(\langle A\rangle_+\overline{\langle A\rangle_+}) \in \mathbb{R}. \tag{40}$$

Replacing the coefficients of $\langle A\rangle_+$ in (40) now by complex (respectively hyperbolic) coefficients

$$\alpha \to \alpha + i\beta, \quad B_{jk} \to B_{jk} + iC_{jk}, \quad F_{jklm} \to F_{jklm} + iG_{jklm}, \tag{41}$$

where $1 \leq j < k \leq 5$, $k < l < m \leq 5$, that is replacing $\langle A\rangle_+ \to A_c = \langle A\rangle_+ + \langle A\rangle_-^* i$ in (40), we instead obtain a complex (hyperbolic) scalar

$$y = A_c\overline{A}_c\, \mathrm{m}_{\bar{3},\bar{4}}(A_c\overline{A}_c), \tag{42}$$

where it is *very important* to notice, that any involution is *not applied* to the complex (or hyperbolic) imaginary unit i. This complex (or hyperbolic) scalar (42) can now be turned into the real determinant z of $A \in Cl(q,p)$ by grade involution gi$(\ldots)$, after isomorphically mapping back $i \to I$,

$$z = \det(A) = y\,\widehat{y} = y\,\mathrm{gi}\,(y)\,, \tag{43}$$

where grade involution maps $I \to \widehat{I} = \mathrm{gi}(I) = -1$. The determinant z yields an expression for the full inverse of $A \in Cl(q,p)$, $p+q = 5$,

$$A^{-1} = \frac{\overline{A}m_{\bar{3},\bar{4}}(A\overline{A})\,\widehat{y}}{z} \tag{44}$$

Note again that, in the above expressions, Clifford conjugation (overline), and the involution $m_{\bar{3},\bar{4}}$ are solely applied with the respect to the complex (or hyperbolic) expression of A in the 16-dimensional basis (39), arising from the even subalgebra isomorphism (34). Clifford conjugation (overline), and the involution $m_{\bar{3},\bar{4}}$, thus do not affect the pseudoscalar I.

As concrete example we take in $Cl(5,0)$ the multivector

$$\begin{aligned} A &= 1 + 2I + (3+4I)e_{12} + (5+6I)e_{1234} \\ &= 1 + 6Ie_{1234} + 3e_{12} + 4Ie_{12} + 5e_{1234} + 2I \\ &= 1 + 6e_5 + 3e_{12} - 4e_{345} + 5e_{1234} + 2e_{12345} \\ &= 1 + 2I + (3+4I)E_1 - (5+6I)E_{123}. \end{aligned} \tag{45}$$

Replacing in (45) $I \to u$, we obtain the isomorphic hyperbolic coefficient multivector A_h in hyperbolic complexified $Cl(0,4) \cong Cl^+(5,0)$, with $E_1, E_{123} \in Cl(0,4)$, as

$$A_h = 1 + 2u + (3+4u)E_1 - (5+6u)E_{123}. \tag{46}$$

Note that here $E_1^2 = e_{12}^2 = -1$, $E_{123}^2 = (-e_{1234})^2 = 1$, and $u^2 = 1$. The hyperbolic scalar determinant of A_h of (46) can be computed step by step via

$$\begin{aligned} A_h\overline{A}_h &= (1 + 2u + (3+4u)E_1 - (5+6u)E_{123}) \\ &\quad \overline{(1 + 2u + (3+4u)E_1 - (5+6u)E_{123})} \\ &= (1 + 2u + (3+4u)E_1 - (5+6u)E_{123}) \\ &\quad (1 + 2u - (3+4u)E_1 - (5+6u)E_{123}) \\ &= 91 + 88u - (34+32u)E_{123}, \end{aligned} \tag{47}$$

and further as hyperbolic scalar

$$\begin{aligned} y &= A_h\overline{A}_h\, \mathrm{m}_{\bar{3},\bar{4}}(A_h\overline{A}_h) \\ &= (91 + 88u - (34+32u)E_{123})(91 + 88u + (34+32u)E_{123}) \\ &= (91+88u)^2 - (34+32u)^2 = 13845 + 13840u. \end{aligned} \tag{48}$$

Finally, after isomorphically mapping back $u \to I$, and by using grade involution $\mathrm{gi}(\ldots) = \widehat{\ldots}$, we can compute the real determinant of $A \in Cl(p,q)$, $p+q=5$, of (45) as

$$z = \det(A) = y\,\widehat{y} = (13845 + 13840I)(13845 - 13840I) = 138425. \qquad (49)$$

And the inverse of A_h of (46) is therefore

$$\begin{aligned} A_h^{-1} &= \frac{\overline{A}_h\, m_{\bar{3},\bar{4}}(A_h\overline{A}_h)\,\widehat{y}}{z} \\ &= \frac{1}{138425}(1 + 2u - (3+4u)E_1 - (5+6u)E_{123}) \\ &\qquad (91 + 88u + (34+32u)E_{123})(13845 - 13840u) \\ &= \frac{1}{138425}\,(-14315 + 13370u + (38395 - 44660u)E_1 \\ &\qquad +(-26530 + 28840u)E_{23} + (9415 - 18270u)E_{123})\,. \end{aligned} \qquad (50)$$

Replacing $u \to I = e_{12345}$, and using $E_1 = e_{12}$, $IE_1 = -e_{345}$, $E_{23} = E_2E_3 = e_{13}e_{14} = -e_{34}$, $IE_{23} = -Ie_{34} = e_{125}$, $E_{123} = E_1E_{23} = -e_{12}e_{34} = -e_{1234}$, $IE_{123} = -Ie_{1234} = -e_5$, we obtain the correct full multivector inverse of $A \in Cl(p,q)$, $p+q=5$, i.e. of (45), as

$$\begin{aligned} A^{-1} &= \frac{1}{138425}(-14315 + 13370I + 38395E_1 - 44660IE_1 - 26530E_{23} \\ &\quad + 28840IE_{23} + 9415E_{123} - 18270IE_{123}) \\ &= \frac{1}{138425}(-14315 + 18270e_5 + 38395e_{12} + 26530e_{34} + 28840e_{125} \\ &\quad + 44660e_{345} - 9415e_{1234} + 13370e_{12345}) \\ &= \frac{1}{27685}(-2863 + 3654e_5 + 7679e_{12} + 5306e_{34} + 5768e_{125} \\ &\quad + 8932e_{345} - 1883e_{1234} + 2674e_{12345}). \end{aligned} \qquad (51)$$

3.3. Multivector inverse for Clifford algebras over 7-dimensional vector spaces from multivector inverse for Clifford algebras over 6-dimensional vector spaces

The isomorphisms (10) for the eight Clifford algebras over $n = p+q = 7$-dimensional vector spaces are the following

$$\begin{aligned} &Cl^+(7,0) \cong Cl^+(0,7) \cong Cl(0,6), \\ &Cl^+(6,1) \cong Cl^+(1,6) \cong Cl(6,0) \cong Cl(1,5), \\ &Cl^+(5,2) \cong Cl^+(2,5) \cong Cl(5,1) \cong Cl(2,4), \\ &Cl^+(4,3) \cong Cl^+(3,4) \cong Cl(4,2) \cong Cl(3,3). \end{aligned} \qquad (52)$$

Moreover the central pseudoscalar $I = e_1e_2e_3e_4e_5e_6e_7 = e_{1234567}$ squares to -1 for $Cl(7,0)$, $Cl(5,2)$, $Cl(3,4)$ and $Cl(1,6)$; while it squares to $+1$ for $Cl(0,7)$, $Cl(2,5)$, $Cl(4,3)$ and $Cl(6,1)$, respectively.

A general multivector $A \in Cl(p,q)$, $p+q=7$, can be represented grade wise as

$$\begin{aligned} A &= \alpha + GI + B + CI + F + HI + J + \beta I \\ &= \alpha + \beta I + B + HI + F + CI + J + GI \\ &= \alpha + B + F + J + (\beta + H + C + G)I \\ &= \langle A\rangle_+ + (\langle A\rangle_- I^{-1})I = \langle A\rangle_+ + \langle A\rangle_-^* I, \end{aligned} \tag{53}$$

with even grade dual of the odd part $\langle A\rangle_-^* = \langle A\rangle_- I^{-1}$, scalars $\alpha, \beta \in \mathbb{R}$, bivectors $B, H \in Cl^2(p,q)$, 4-vectors $F, C \in Cl^4(p,q)$, and 6-vectors $J, G \in Cl^6(p,q)$ such that vector $GI \in \mathbb{R}^{p,q}$, trivector $CI \in Cl^3(p,q)$, and 5-vector $HI \in Cl^5(p,q)$. Expressed as above, $A \in Cl(p,q)$, $p+q=7$, takes the form of a complex (or hyperbolic) scalar[8] $\alpha + \beta i$ plus a complex (or hyperbolic) bivector $B+Hi$ plus a complex (or hyperbolic) 4-vector $F+Ci$ plus a complex (or hyperbolic) 6-vector $J+Gi$. And both $\langle A\rangle_+, \langle A\rangle_-^* \in Cl^+(p,q)$. Via the isomorphisms (52), we can isomorphically map the even part[9] $\langle A\rangle_+ = \alpha + B + F + J$ to a Clifford algebra $Cl(p,q-1)$ or $Cl(q,p-1)$ of a six-dimensional ($n' = p+q-1 = 7-1 = 6$) vector space.

Assuming any known inverse formula for multivectors of non-zero determinant in $Cl(p,q-1)$ or $Cl(q,p-1)$, we can compute the inverse of the even grade part $\langle A\rangle_+$. The determinant of $\langle A\rangle_+$ will be real, but if we replace the coefficients by complex (respectively hyperbolic) numbers, the determinant will be complex (or hyperbolic). The product of this determinant with its complex (or hyperbolic) conjugate will produce a real number, which is identical to the determinant of the full multivector $A \in Cl(p,q)$, $p+q=7$, and therefore leads (if non-zero) to an inverse for the full multivector $A \in Cl(p,q)$, $p+q=7$. The even $2^6 = 64$-dimensional subalgebra $Cl^+(p,q)$ of $Cl(p,q)$, $p+q=7$, has the following 64-dimensional even grade element basis

$$\begin{aligned} &\{1, \\ &E_1 = e_{12}, E_2 = e_{13}, E_3 = e_{14}, E_4 = e_{15}, E_5 = e_{16}, E_6 = e_{17}, \\ &e_{23}, e_{24}, e_{25}, e_{26}, e_{27}, e_{34}, e_{35}, e_{36}, e_{37}, e_{45}, e_{46}, e_{47}, e_{56}, e_{57}, e_{67}, \\ &e_{1234}, e_{1235}, e_{1236}, e_{1237}, e_{1245}, e_{1246}, e_{1247}, e_{1256}, e_{1257}, e_{1267}, \\ &e_{1345}, e_{1346}, e_{1347}, e_{1356}, e_{1357}, e_{1367}, e_{1456}, e_{1457}, e_{1467}, e_{1567}, \\ &e_{2345}, e_{2346}, e_{2347}, e_{2356}, e_{2357}, e_{2367}, e_{2456}, e_{2457}, e_{2467}, e_{2567}, \\ &e_{3456}, e_{3457}, e_{3467}, e_{3567}, e_{4567}, \\ &e_{123456}, e_{123457}, e_{123467}, e_{123567}, e_{124567}, e_{134567}, \\ &e_{234567}\}, \end{aligned} \tag{54}$$

[8] Here we apply the isomorphism to the complexified even subalgebra $Cl^+(p,q)$, where i now stands both for the complex imaginary unit (case $I^2 = -1$) and the hyperbolic unit (case $I^2 = +1$).

[9] Correspondingly, the full multivector A is mapped to the complexified (hyperbolic complexified) isomorphic algebra.

which can be rewritten, using the isomorphisms (52), even basis blade by even basis blade, in terms of the isomorphic lower dimensional Clifford algebra, generated from the basis $\{E_1, E_2, E_3, E_4, E_5, E_6\}$, as ($s = -e_1^2 = \pm 1$, depending on the square of e_1)

$$
\begin{aligned}
&\{1,\\
&E_1, E_2, E_3, E_4, E_5, E_6\\
&sE_{12}, sE_{13}, sE_{14}, sE_{15}, sE_{16}, sE_{23}, sE_{24}, sE_{25}, sE_{26}, sE_{34}, sE_{35}, sE_{36},\\
&sE_{45}, sE_{46}, sE_{56},\\
&sE_{123}, sE_{124}, sE_{125}, sE_{126}, sE_{134}, sE_{135}, sE_{136}, sE_{145}, sE_{146}, sE_{156},\\
&sE_{234}, sE_{235}, sE_{236}, sE_{245}, sE_{246}, sE_{256}, sE_{345}, sE_{346}, sE_{356}, sE_{456},\\
&E_{1234}, E_{1235}, E_{1236}, E_{1245}, E_{1246}, E_{1256}, E_{1345}, E_{1346}, E_{1356}, E_{1456},\\
&E_{2345}, E_{2346}, E_{2356}, E_{2456}, E_{3456},\\
&E_{12345}, E_{12346}, E_{12356}, E_{12456}, E_{13456}, E_{23456},\\
&sE_{123456}\}.
\end{aligned}
\tag{55}
$$

That means, that the even multivector $\langle A\rangle_+ = \alpha + B + F + J$ can be represented isomorphically in two ways

$$
\begin{aligned}
\langle A\rangle_+ = \alpha 1&\\
&+ B_{12}e_{12} + B_{13}e_{13} + B_{14}e_{14} + B_{15}e_{15} + B_{16}e_{16} + B_{17}e_{17} + B_{23}e_{23}\\
&+ B_{24}e_{24} + B_{25}e_{25} + B_{26}e_{26} + B_{27}e_{27} + B_{34}e_{34} + B_{35}e_{35} + B_{36}e_{36}\\
&+ B_{37}e_{37} + B_{45}e_{45} + B_{46}e_{46} + B_{47}e_{47} + B_{56}e_{56} + B_{57}e_{57} + B_{67}e_{67}\\
&+ F_{1234}e_{1234} + F_{1235}e_{1235} + F_{1236}e_{1236} + F_{1237}e_{1237} + F_{1245}e_{1245}\\
&+ F_{1246}e_{1246} + F_{1247}e_{1247} + F_{1256}e_{1256} + F_{1257}e_{1257} + F_{1267}e_{1267}\\
&+ F_{1345}e_{1345} + F_{1346}e_{1346} + F_{1347}e_{1347} + F_{1356}e_{1356} + F_{1357}e_{1357}\\
&+ F_{1367}e_{1367} + F_{1456}e_{1456} + F_{1457}e_{1457} + F_{1467}e_{1467} + F_{1567}e_{1567}\\
&+ F_{2345}e_{2345} + F_{2346}e_{2346} + F_{2347}e_{2347} + F_{2356}e_{2356} + F_{2357}e_{2357}\\
&+ F_{2367}e_{2367} + F_{2456}e_{2456} + F_{2457}e_{2457} + F_{2467}e_{2467} + F_{2567}e_{2567}\\
&+ F_{3456}e_{3456} + F_{3457}e_{3457} + F_{3467}e_{3467} + F_{3567}e_{3567} + F_{4567}e_{4567}\\
&+ J_{123456}e_{123456} + J_{123457}e_{123457} + J_{123467}e_{123467} + J_{123567}e_{123567}\\
&+ J_{124567}e_{124567} + J_{134567}e_{134567}\\
&+ J_{234567}e_{234567}
\end{aligned}
\tag{56}
$$

or alternatively as

$$\begin{aligned}\langle A\rangle_+ = {}& \alpha 1 + B_{12}E_1 + B_{13}E_2 + B_{14}E_3 + B_{15}E_4 + B_{16}E_5 + B_{17}E_6 \\ &+ sB_{23}E_{12} + sB_{13}E_{24} + sB_{25}E_{14} + sB_{26}E_{15} + sB_{27}E_{16} + sB_{34}E_{23} \\ &+ sB_{35}E_{24} + sB_{36}E_{25} + sB_{37}E_{26} + sB_{45}E_{34} + sB_{46}E_{35} + sB_{47}E_{36} \\ &+ sB_{56}E_{45} + sB_{57}E_{46} + sB_{67}E_{56} \\ &+ sF_{1234}E_{123} + sF_{1235}E_{124} + sF_{1236}E_{125} + sF_{1237}E_{126} + sF_{1245}E_{134} \\ &+ sF_{1246}E_{135} + sF_{1247}E_{136} + sF_{1256}E_{145} + sF_{1257}E_{146} + sF_{1267}E_{156} \\ &+ sF_{1345}E_{234} + sF_{1346}E_{235} + sF_{1347}E_{236} + sF_{1356}E_{245} + sF_{1357}E_{246} \\ &+ sF_{1367}E_{256} + sF_{1456}E_{345} + sF_{1457}E_{346} + sF_{1467}E_{356} + sF_{1567}E_{456} \\ &+ F_{2345}E_{1234} + F_{2346}E_{1235} + F_{2347}E_{1236} + F_{2356}E_{1245} + F_{2357}E_{1246} \\ &+ F_{2367}E_{1256} + F_{2456}E_{1345} + F_{2457}E_{1346} + F_{2467}E_{1356} + F_{2567}E_{1456} \\ &+ F_{3456}E_{2345} + F_{3457}E_{2346} + F_{3467}E_{2356} + F_{3567}E_{2456} + F_{4567}E_{3456} \\ &+ J_{123456}E_{12345} + J_{123457}E_{12346} + J_{123467}E_{12356} + J_{123567}E_{12456} \\ &+ J_{124567}E_{13456} + J_{134567}E_{23456} \\ &+ sJ_{234567}E_{123456}. \end{aligned} \tag{57}$$

With respect to the 64-dimensional blade basis (55) of $Cl(p, q-1)$ or $Cl(q, p-1)$, $p+q-1 = 6$, we now compute from $\langle A\rangle_+$ of (57) the real scalar (see [2])

$$\begin{aligned}&\langle A\rangle_+\widetilde{\langle A\rangle_+}\Big(\tfrac{1}{3}\langle A\rangle_+\widetilde{\langle A\rangle_+}\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(\langle A\rangle_+\widetilde{\langle A\rangle_+}\langle A\rangle_+\widetilde{\langle A\rangle_+}) \\ &\quad+\tfrac{2}{3}\mathrm{m}_{\bar{4}}(\mathrm{m}_{\bar{4}}(\langle A\rangle_+\widetilde{\langle A\rangle_+})\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(\mathrm{m}_{\bar{4}}(\langle A\rangle_+\widetilde{\langle A\rangle_+})\mathrm{m}_{\bar{4}}(\langle A\rangle_+\widetilde{\langle A\rangle_+})))\Big) \in \mathbb{R}.\end{aligned} \tag{58}$$

Replacing the coefficients of $\langle A\rangle_+$ in (58) now by complex (respectively hyperbolic) coefficients

$$\begin{aligned}&\alpha \to \alpha + i\beta, \quad B_{jk} \to B_{jk} + iH_{jk}, \quad F_{jklm} \to F_{jklm} + iC_{jklm} \\ &J_{jklmst} \to J_{jklmst} + iG_{jklmst},\end{aligned} \tag{59}$$

where $1 \le j < k \le 7$, $k < l < m \le 7$, $m < s < t \le 7$, that is replacing $\langle A\rangle_+ \to A_c = \langle A\rangle_+ + \langle A\rangle_-^* i$ in (58), we instead obtain a complex (hyperbolic) scalar

$$\begin{aligned}y = A_c\widetilde{A}_c &\Big(\tfrac{1}{3}A_c\widetilde{A}_c\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(A_c\widetilde{A}_cA_c\widetilde{A}_c) \\ &+\tfrac{2}{3}\mathrm{m}_{\bar{4}}(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)))\Big),\end{aligned} \tag{60}$$

where it is *very important* to notice, that any involution is *not applied* to the complex (hyperbolic) unit i. Note, that for obtaining the scalar y of (60), only a *single* addition of multivector products is needed. This complex (or hyperbolic) scalar (60) can now be turned into the real determinant z of $A \in Cl(q, p)$ by grade involution $\mathrm{gi}(\ldots)$, after isomorphically mapping back $i \to I$,

$$z = \det(A) = y\,\widehat{y} = y\,\mathrm{gi}(y), \tag{61}$$

where grade involution maps $I \to \widehat{I} = \mathrm{gi}(I) = -I$. The determinant z yields an expression[10] for the full inverse of $A \in Cl(q,p)$, $n = p+q = 7$, if we simultaneously apply the isomorphism for all basis elements (55) back from $Cl(0,6)$ to (54) in $Cl^+(7,0)$,

$$A^{-1} = \frac{1}{z}\widetilde{A}\Big(\tfrac{1}{3}A\widetilde{A}\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(A\widetilde{A}A\widetilde{A}) + \tfrac{2}{3}\mathrm{m}_{\bar{4}}(\mathrm{m}_{\bar{4}}(A\widetilde{A})\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(\mathrm{m}_{\bar{4}}(A\widetilde{A})\,\mathrm{m}_{\bar{4}}(A\widetilde{A})))\Big)\,\mathrm{gi}(y)\,. \quad (62)$$

Note again that, in the above expressions, reversion (tilde), and the involutions $m_{\bar{4}}$, $m_{\bar{1},\bar{4},\bar{5}}$, are solely applied with the respect to the complex (or hyperbolic) expression A_c for A in the 64-dimensional basis (55), arising from the even subalgebra isomorphism[11] (52). Reversion (tilde), and the involutions $m_{\bar{4}}$, $m_{\bar{1},\bar{4},\bar{5}}$, thus do not affect the pseudoscalar I.

As concrete example for $n = 7$ we take the following multivector $A \in Cl(7,0)$,

$$\begin{aligned} A &= 1 + 2I + (3+4I)e_{12} + (5+6I)e_{1234} + (7+8I)e_{123456} \\ &= 1 + 8Ie_{123456} + 3e_{12} + 6Ie_{1234} + 5e_{1234} + 4Ie_{12} + 7e_{123456} + 2I \\ &= 1 - 8e_7 + 3e_{12} + 6e_{567} + 5e_{1234} - 4e_{34567} + 7e_{123456} + 2e_{1234567} \\ &= 1 + 2I + (3+4I)E_1 - (5+6I)E_{123} + (7+8I)E_{12345}. \end{aligned} \quad (63)$$

Replacing in (63) $I \to i$, we obtain the isomorphic complex coefficient multivector A_c in complexified $Cl(0,6) \cong Cl^+(7,0)$, with $E_1, E_{123}, E_{12345} \in Cl(0,6)$, as

$$A_c = 1 + 2i + (3+4i)E_1 - (5+6i)E_{123} + (7+8i)E_{12345}. \quad (64)$$

Note that here $E_1^2 = e_{12}^2 = -1$, $E_{123}^2 = (-e_{1234})^2 = 1$, $E_{12345}^2 = e_{123456}^2 = -1$, $E_{2345}^2 = +1$, and $i^2 = -1$. The complex scalar determinant of A_c in (64) can be computed step by step as follows. We first compute

$$\begin{aligned} A_c\widetilde{A}_c &= (1 + 2i + (3+4i)E_1 - (5+6i)E_{123} + (7+8i)E_{12345}) \\ &\quad (1 + 2i + (3+4i)E_1 - (5+6i)E_{123} + (7+8i)E_{12345})^\sim \\ &= (1 + 2i + (3+4i)E_1 - (5+6i)E_{123} + (7+8i)E_{12345}) \\ &\quad (1 + 2i + (3+4i)E_1 + (5+6i)E_{123} + (7+8i)E_{12345}) \\ &= 30 - 192i + (-10+20i)E_1 + (22-104i)E_{2345} + (-18+44i)E_{12345}. \end{aligned} \quad (65)$$

Next we compute

$$\begin{aligned} &\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c) \\ &= 30 - 192i + (-10+20i)E_1 - (22-104i)E_{2345} + (-18+44i)E_{12345}. \end{aligned} \quad (66)$$

[10] We choose below the notation $\mathrm{gi}(y) = \widehat{y}$ for the grade involution, convenient for long expressions as in (75).

[11] To be able to correctly apply all necessary involutions, actually appears to be the main necessity of switching to the even subalgebra basis (55).

Squaring $A_c\widetilde{A}_c$ we obtain

$$\begin{aligned}
&A_c\widetilde{A}_cA_c\widetilde{A}_c \\
&= (30-192i+(-10+20i)E_1+(22-104i)E_{2345}+(-18+44i)E_{12345}) \\
&\quad (30-192i+(-10+20i)E_1+(22-104i)E_{2345}+(-18+44i)E_{12345}) \\
&= -44384-14112i+(15440+10720i)E_1 \\
&\quad +(-37216-13088i)E_{2345}+(19536E_{12345}+12512i)E_{12345},
\end{aligned} \tag{67}$$

and squaring $\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)$ we likewise obtain

$$\begin{aligned}
&\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\,\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c) \\
&= (30-192i+(-10+20i)E_1-(22-104i)E_{2345}+(-18+44i)E_{12345}) \\
&\quad (30-192i+(-10+20i)E_1-(22-104i)E_{2345}+(-18+44i)E_{12345}) \\
&= -44384-14112i-(1280+640i)E_1 \\
&\quad +(40016+16288i)E_{2345}+(12096+6592i)E_{12345}.
\end{aligned} \tag{68}$$

We further compute

$$\begin{aligned}
&A_c\widetilde{A}_c\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(A_c\widetilde{A}_cA_c\widetilde{A}_c) \\
&= (30-192i+(-10+20i)E_1+(22-104i)E_{2345}+(-18+44i)E_{12345}) \\
&\quad \mathrm{m}_{\bar{1},\bar{4},\bar{5}}(-44384-14112i+(15440+10720i)E_1 \\
&\qquad +(-37216-13088i)E_{2345} \\
&\qquad +(19536E_{12345}+12512i)E_{12345}) \\
&= (30-192i+(-10+20i)E_1+(22-104i)E_{2345}+(-18+44i)E_{12345}) \\
&\quad (-44384-14112i-(15440+10720i)E_1-(-37216-13088i)E_{2345} \\
&\qquad -(19536E_{12345}+12512i)E_{12345}) \\
&= -3132096+5351808i+(-4772160+5054720i)E_1 \\
&\quad -(9920+1695360i)E_{2345}+(-3657024+3660032i)E_{12345},
\end{aligned} \tag{69}$$

and similarly

$$\begin{aligned}
&\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\,\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)) \\
&= (30-192i+(-10+20i)E_1-(22-104i)E_{2345}+(-18+44i)E_{12345}) \\
&\quad \mathrm{m}_{\bar{1},\bar{4},\bar{5}}\,(-44384-14112i-(1280+640i)E_1+(40016+16288i)E_{2345} \\
&\qquad +(12096+6592i)E_{12345}) \\
&= (30-192i+(-10+20i)E_1-(22-104i)E_{2345}+(-18+44i)E_{12345}) \\
&\quad (-44384-14112i+(1280+640i)E_1-(40016+16288i)E_{2345} \\
&\qquad -(12096+6592i)E_{12345}) \\
&= -1948896+4689408i+(3276000-3553600i)E_1 \\
&\quad +(-2085280+3020160i)E_{2345}+(422496-92608i)E_{12345}.
\end{aligned} \tag{70}$$

Therefore, negating grade four we find

$$
\begin{aligned}
&\mathrm{m}_{\bar{4}}\left(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c))\right)\\
&= -1948896 + 4689408i + (3276000 - 3553600i)E_1\\
&\quad + (2085280 - 3020160i)E_{2345} + (422496 - 92608i)E_{12345}. \qquad (71)
\end{aligned}
$$

Next we compute

$$
\begin{aligned}
&\tfrac{1}{3}A_c\widetilde{A}_c\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(A_c\widetilde{A}_cA_c\widetilde{A}_c) + \tfrac{2}{3}\mathrm{m}_{\bar{4}}(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)))\\
&= \tfrac{1}{3}\,(-3132096 + 5351808i + (-4772160 + 5054720i)E_1\\
&\qquad -(9920 + 1695360i)E_{2345} + (-3657024 + 3660032i)E_{12345})\\
&\quad + \tfrac{2}{3}\,(-1948896 + 4689408i + (3276000 - 3553600i)E_1\\
&\qquad +(2085280 - 3020160i)E_{2345} + (422496 - 92608i)E_{12345})\\
&= -2343296 + 4910208i + (593280 - 684160i)E_1\\
&\quad + (1386880 - 2578560i)E_{2345} + (-937344 + 1158272i)E_{12345}. \qquad (72)
\end{aligned}
$$

Then, we get the complex scalar y as

$$
\begin{aligned}
y &= A_c\widetilde{A}_c\left(\tfrac{1}{3}A_c\widetilde{A}_c\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(A_c\widetilde{A}_cA_c\widetilde{A}_c)\right.\\
&\qquad\left.+\tfrac{2}{3}\mathrm{m}_{\bar{4}}(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)))\right)\\
&= (30 - 192i + (-10 + 20i)E_1 + (22 - 104i)E_{2345} + (-18 + 44i)E_{12345})\\
&\quad (-2343296 + 4910208i + (593280 - 684160i)E_1\\
&\qquad +(1386880 - 2578560i)E_{2345} + (-937344 + 1158272i)E_{12345}.)\\
&= 661143552 + 439640064i. \qquad (73)
\end{aligned}
$$

Moreover, after isomorphically mapping back $i \to I$, and by using grade involution $\mathrm{gi}(\ldots) = \widehat{\ldots}$, we can compute the real determinant of our $A \in Cl(7,0)$, i.e. of (63), as

$$
\begin{aligned}
z &= \det(A) = y\widehat{y} = (661143552 + 439640064I)(661143552 - 439640064I)\\
&= 630394182225100800. \qquad (74)
\end{aligned}
$$

And the inverse of A_c of (64), is therefore with real determinant $z = \det(A)$ of (74), and complex scalar y of (73),

$$\begin{aligned}
A_c^{-1} &= \frac{1}{z}\widetilde{A}_c\Big(\tfrac{1}{3}A_c\widetilde{A}_c\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(A_c\widetilde{A}_cA_c\widetilde{A}_c) \\
&\qquad +\tfrac{2}{3}\mathrm{m}_{\bar{4}}(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)\mathrm{m}_{\bar{4}}(A_c\widetilde{A}_c)))\Big)\,\widehat{y} \\
&= \Big(1+2i+(3+4i)E_1-(5+6i)E_{123}+(7+8i)E_{12345}\Big) \\
&\quad \Big(-2343296+4910208i+(593280-684160i)E_1 \\
&\quad +(1386880-2578560i)E_{2345}+(-937344+1158272i)E_{12345}\Big) \\
&\quad \mathrm{gi}\big(661143552+439640064i\big)\frac{1}{630394182225100800} \\
&= \frac{1}{630394182225100800}\Big(-874159543418880-92043073290240i \\
&\quad +(-23804470933585920+27958127744778240i)E_{12345} \\
&\quad +(3239175106068480-3198061888143360i)E_1 \\
&\quad +(3109867062558720-1579168211927042i)E_{2345} \\
&\quad +(-4736241279959039+3017033546465278i)E_{23} \\
&\quad +(-7619749069455360+5226413209681920i)E_{45} \\
&\quad +(-22612009376808960+25039610519224320i)E_{123} \\
&\quad +(-12803600406282240+12872900731207680i)E_{145}\Big).
\end{aligned} \tag{75}$$

Replacing $i \to I = e_{1234567}$, and using $E_1 = e_{12}$, $IE_1 = -e_{34567}$, $E_{23} = E_2E_3 = e_{13}e_{14} = -e_{34}$, $IE_{23} = -Ie_{34} = e_{12567}$, $E_{123} = E_1E_{23} = -e_{12}e_{34} = -e_{1234}$, $IE_{123} = -Ie_{1234} = -e_{567}$, $E_{45} = -e_{56}$, $IE_{45} = -Ie_{56} = e_{12347}$, $E_{145} = -e_{1256}$, $IE_{145} = -Ie_{1256} = -e_{347}$, $E_{2345} = e_{3456}$, $IE_{2345} = e_{127}$, $E_{12345} = e_{123456}$, $IE_{12345} = Ie_{123456} = -e_7$, we finally obtain the correct full multivector inverse of $A \in Cl(7,0)$, i.e. of (63), as

$$\begin{aligned}
A^{-1} = \frac{1}{630394182225100800}\Big(&-874159543418880-27958127744778240e_7 \\
&+3239175106068480e_{12}+4736241279959039e_{34} \\
&+7619749069455360e_{56}-1579168211927042e_{127} \\
&-12872900731207680e_{347}-25039610519224320e_{567} \\
&+22612009376808960e_{1234}+12803600406282240e_{1256} \\
&+3109867062558720e_{3456}+5226413209681920e_{12347} \\
&+3017033546465278e_{12567}+3198061888143360e_{34567} \\
&-23804470933585920e_{123456}-92043073290240e_{1234567}\Big).
\end{aligned} \tag{76}$$

4. Multivector inverse for Clifford algebras over $2m+1$-dimensional vector spaces from multivector inverse for Clifford algebras over $2m$-dimensional vector spaces

The general case proceeds in the same way as the examples for $n = 3, 5, 7$ vector space dimensions, given above. Assume that for Clifford algebras $Cl(p', q')$, $n' = 2m = p' + q'$, the algebraic expression for computing a general multivector inverse[12] is known

$$X^{-1} = \frac{f(X)}{\det X}, \quad \forall X \in Cl(p', q'), \;\; n' = 2m = p' + q', \;\; \det(X) \neq 0, \qquad (77)$$

involving the determinant $\det(X) = Xf(X)$ as denominator of X^{-1}, the multivector $f(X)$ being a product of involutions of X (or sum of products of involutions).

For illustration, we give the expressions for $f(X)$ used in the examples of Section 3, based on (24), (40) (without the leftmost leading factor X), and (58) (without the leftmost leading factor X):

$$\begin{aligned}
&\widetilde{X} \quad \text{for} \quad n' = 2, \\
&\overline{X}\mathrm{m}_{\bar{3},\bar{4}}(X\overline{X}) \quad \text{for} \quad n' = 4, \\
&\widetilde{X}\left(\tfrac{1}{3}X\widetilde{X}\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(X\widetilde{X}X\widetilde{X}) + \tfrac{2}{3}\mathrm{m}_{\bar{4}}(\mathrm{m}_{\bar{4}}(X\widetilde{X})\,\mathrm{m}_{\bar{1},\bar{4},\bar{5}}(\mathrm{m}_{\bar{4}}(X\widetilde{X})\mathrm{m}_{\bar{4}}(X\widetilde{X})))\right) \\
&\qquad \text{for} \quad n' = 6. \qquad (78)
\end{aligned}$$

We can always split a multivector $A \in Cl(p, q)$, $n = 2m + 1 = p + q = p' + q' + 1$, into its even and odd parts, and rewrite the odd part as the dual of an even multivector

$$A = \langle A\rangle_+ + \langle A\rangle_- = \langle A\rangle_+ + \langle A\rangle_-^* I, \qquad (79)$$

where the central pseudoscalar $I = e_1e_2 \ldots e_n$ will either have negative or positive square. The even subalgebra $Cl^+(p, q)$, $n = 2m + 1 = p + q + 1$, is by the isomorphism (10), isomorphic to some Clifford algebra $Cl(p', q')$, $n' = n - 1 = 2m = p' + q'$. By the above assumption (77), the multivector inverse for $\langle A\rangle_+$, $\det(\langle A\rangle_+) \neq 0$, is known in $Cl(p', q')$, $n' = 2m = p' + q'$,

$$\langle A\rangle_+^{-1} = \frac{f(\langle A\rangle_+)}{\det(\langle A\rangle_+)}, \qquad \det(\langle A\rangle_+) = \langle A\rangle_+ f(\langle A\rangle_+). \qquad (80)$$

Our next step is always to replace the real even multivector $\langle A\rangle_+$ in (80) by a complex even multivector $\langle A\rangle_+ \to A_c = \langle A\rangle_+ + \langle A\rangle_-^* i$, $i^2 = -1$, in the case of $I^2 = -1$, or alternatively by a hyperbolic even multivector $\langle A\rangle_+ \to A_h = \langle A\rangle_+ + \langle A\rangle_-^* u$, $u^2 = +1$, in the case of $I^2 = +1$. This will change the real

[12] Experience, algebraic, symbolic and numerical computations for $n' < 7$ have shown, that there is good reason to *conjecture*, that these general multivector inverses have algebraic expressions, which depend only on the dimension of the vector space, but not on the details of the signature indexes p, q, see [3, 12, 6, 2].

determinant $\det(\langle A\rangle_+)$ to a complex (or hyperbolic) determinant

$$\begin{aligned}\det(\langle A\rangle_+) \;\to\; & y = \det(A_{c,h}) \\ & = A_{c,h} f(A_{c,h}) \in \mathbb{R}\oplus\mathbb{R}i \text{ or } \mathbb{R}\oplus\mathbb{R}u,\end{aligned} \tag{81}$$

where it is important to note, that any involutions involved in computing $f(A_{c,h})$, are *not applied* to the complex imaginary unit i (or the hyperbolic unit u). With the help of complex (or hyperbolic) conjugation, we can turn the complex (or hyperbolic) determinant $y = \det(A_{c,h})$ of (81) into a higher order real determinant $z = \det(A)$ of $A \in Cl(p,q)$, $n = 2m+1 = p+q+1$,

$$z = \det(A) = \begin{cases} \det(A_c)\,\mathrm{cc}(\det(A_c)), & I^2 = -1, \\ \det(A_h)\,\mathrm{hc}(\det(A_h)), & I^2 = +1, \end{cases} \tag{82}$$

where, after back replacement $i \to I$ ($u \to I$), isomorphically mapping $A_{c,h} \to A = \langle A\rangle_+ + \langle A\rangle_-^* I$, the grade involution $\mathrm{gi}(\ldots)$ precisely has the effect of complex (or hyperbolic) conjugation, i.e. $\mathrm{gi}(I) = -I$. By construction $z = \det(A)$ will have the multivector $A_{c,h}$ itself as first factor $A_{c,h}$ on the left, see (81) and (82). Removing this first factor, we get for $\det(A) \neq 0$, the final expression for the *inverse* of $A_{c,h}$, as

$$A_{c,h}^{-1} = \begin{cases} f(A_c)\,\mathrm{cc}(y)/z, & I^2 = -1, \\ f(A_h)\,\mathrm{hc}(y)/z, & I^2 = +1. \end{cases} \tag{83}$$

This then gives the full inverse for $A \in Cl(p,q)$, $n = n'+1 = p+q = 2m+1$, for $z = \det(A) \neq 0$, by replacing $i \to I$ ($u \to I$), and back substituting the isomorphism (10) from the Clifford algebra basis $Cl(p',q')$, $n' = 2m = p'+q' = n-1$ to the even subalgebra of the Clifford algebra $Cl^+(p,q)$, $n = p+q = n+1$.

$$A^{-1} = \frac{f(A)\,\widehat{y}}{z}, \tag{84}$$

Note again that, in the above expression, any involutions in $f(A)$, are solely applied with the respect to the isomorphic complex (or hyperbolic) expression $A_{c,h}$ for A in the $2^{n'} = 2^{(n-1)}$-dimensional basis, arising from the even subalgebra isomorphism (10). The involutions applied in $f(\ldots)$, thus do not affect the pseudoscalar I. If the determinant $\det(A)$ in (82) turns out to be zero, then $A \in Cl(p,q)$, $n = 2m+1 = p+q+1$, is a *zero divisor* and has no inverse. As pointed out in Section 2, the inverse multivector of (83), is at the same time always left- and right inverse.

If in Clifford algebras $Cl(p',q')$, $n' = 2m = p'+q'$, the algebraic expression for computing a general multivector inverse is instead of (77) given as

$$X^{-1} = \frac{g(X)}{\det X}, \quad \forall X \in Cl(p',q'),\;\; n' = 2m = p'+q',\;\; \det(X) \neq 0, \tag{85}$$

involving the determinant $\det(X) = g(X)X$ as denominator of X^{-1}, the multivector $g(X)$ being a product of involutions of X (or sum of products of involutions), then the corresponding expression for the inverse of $A \in Cl(p,q)$, $n = 2m+1 = p+q+1$ becomes

$$\begin{aligned} A^{-1} &= \frac{\widehat{y}\, g(A)}{z}, \\ y &= \det(A_{c,h}) = g(A_{c,h})A_{c,h}, \\ z &= \det(A) = \begin{cases} \mathrm{cc}(\det(A_c))\det(A_c), & I^2 = -1, \\ \mathrm{hc}(\det(A_h))\det(A_h), & I^2 = +1. \end{cases} \end{aligned} \tag{86}$$

5. Conclusion

Assuming known algebraic expressions for multivector inverses in any Clifford algebra over an even dimensional vector space $\mathbb{R}^{p',q'}$, $n' = p'+q' = 2m$, we were able to derive a closed algebraic expression for the multivector inverse over vector spaces one dimension higher, namely over $\mathbb{R}^{p,q}$, $n = p+q = p'+q'+1 = 2m+1$. Explicit examples were given for dimensions $n' = 2,4,6$, and the resulting inverses for $n = n'+1 = 3,5,7$. The general result for $n = 7$ appears to be the first ever reported closed algebraic expression for a multivector inverse in Clifford algebras $Cl(p,q)$, $n = p+q = 7$, only involving a *single* addition in forming the multivector determinant, as in the denominator of (62). The compact form of this expression is advantageous for increasing speed and accuracy of numeric computations, and speed of symbolic computations. Furthermore, the octonion product can be expressed in $Cl(0,7)$ as product of paravectors (scalars plus vectors) multiplied by $(1 - e_{124} - e_{235} - e_{346} - e_{457} - e_{561} - e_{672} - e_{713})$, finally followed by projection to paravectors (grades zero and one), see Chapter 23.3 of [8]. The compact expression for the multivector inverse also valid in $Cl(0,7)$, which we have derived, may therefore also prove useful for symbolic and numeric octonion algebra computations.

With the result of this paper, it becomes possible to make a step from compact algebraic expressions for multivector inverses in Clifford algebras over even ($n = 2m$) dimensional spaces to similarly compact expressions for inverses of multivectors in Clifford algebras over spaces one dimension higher ($n = 2m+1$). It would be most desirable to find a similar step from multivector inverses in Clifford algebras over odd dimensional vector spaces to multivector inverses in Clifford algebras over vector spaces one dimension higher (and thus even dimensional). Then one would have a complete inductive ladder for efficiently computing multivector inverses in Clifford algebras of any dimension.

We thank R. Abłamowicz for pointing out that with the help of the characteristic polynomial of a multivector M the inverse can be computed as well, which is e.g. implemented in the Clifford package for Maple [1]. For this method first the characteristic polynomial coefficients have to be computed from $\det(\lambda - M)$, in the case of square matrices $\det(\lambda E - A_M)$, where E is the unit matrix, and A_M is the matrix isomorphic to M in an Nth order square

matrix representation. Clifford algebras $Cl(p,q)$, $p+q=7$ are isomorphic[13] to complex 8×8 square matrices, pairs of real 8×8 matrices or pairs of quaternionic 4×4 matrices, see Table 1 in Chapter 16.4 of [8]. The constant coefficient of the characteristic polynomial is $\pm \det(M)$, the other terms form a polynomial $-MP_{N-1}(M)$, where $P_{N-1}(M)$ is a polynomial of the order $N-1$. Therefore $P_{N-1}(M)$ divided by $\det(M)$ gives the inverse M^{-1}. In the case of $Cl(p,q)$, $p+q=7$, N would be at least 8, the characteristic polynomial would first have to be computed, and all powers of M up to the order of $N-1=7$. From a numerical viewpoint of computational cost and accuracy, the method (86) for $n'=6$ also involves the computation of the determinant, but we can avoid to first compute the characteristic polyonomial coefficients and their use for obtaining the multivector or isomorphic matrix polynomial $P_{N-1}(M)$. Moreover, the conjugations we use only involve switching signs of certain grade parts, which is very fast and introduces no alteration to numerical values. We therefore expect that even in different software environments our approach for the case $Cl(p,q)$, $p+q=7$, should be faster and numerically more accurate.

E. H. would like to urge readers to apply the knowledge communicated in this paper in accordance with the Creative Peace License [5].

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