The Friedmann-Lemaître-Robertson-Walker Metric with a Constant Curvature Scalar

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Abstract

In the complete absence of ordinary matter, the Friedmann-Lemaître-Robertson-Walker (FLRW) metric in the dark energy-dominated era of late cosmological evolution features a constant curvature scalar (the Ricci scalar R) that is proportional to the cosmological constant Λ. Indeed, in the distant future the universe is expected to approach a pure de Sitter spacetime, in which stray high-entropy photons make up the bulk of the universe's energy. However, similar conditions may have also existed immediately following the Big Bang, when radiation from highly-relativistic neutrinos and photons comprised the majority of the energy in the expanding universe. If true, then it is plausible that the curvature scalar R was also constant in the radiation-dominated era of the early universe. This motivates the possibility of a metric that involves a constant R in the intervening matter-dominated era, which existed between some 50,000 years after the Big Bang and roughly four billion years ago, when dark energy began to overcome the dominance of gravitating matter.

To investigate the consequence of a constant universal curvature scalar, the FLRW metric is derived from Einstein's field equations featuring both Λ and a constant R in the presence of non-interacting matter having a proper density $\rho(t)$ and pressure $P(\rho)$. The results indicate that the formalism is valid only for a relativistic fluid with the dimensionless equation of state parameters $\omega = -1$ and $\omega = 1/3$, corresponding to the very early and very late universe, respectively.

1. The Friedmann-Lemaître-Robertson-Walker (FLRW) Spacetime

In the interests of brevity, in the following we will simply summarize the equations presented by Adler et al. in the derivation of the FLRW metric. The lower- and upper-case metric tensors used in that approach are

$$g_{\mu\nu} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{g+f} & 0 & 0 \\ 0 & 0 & -r^2 e^{g+f} & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \, e^{g+f} \end{Bmatrix}, \quad g^{\mu\nu} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{-(g+f)} & 0 & 0 \\ 0 & 0 & -\frac{e^{-(g+f)}}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{e^{-(g+f)}}{r^2 \sin^2\theta} \end{Bmatrix}$$

where e^g is a function only of the time t and e^f is a function only of the radial parameter r. The corresponding FLRW metric is then expressed as

$$ds^2 = c^2 dt^2 - e^g e^f \left(dr^2 + +r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right)$$

where perfect isotropy and homogeneity are assumed. The energy-momentum tensor $T_{\mu\nu}$ given by

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2} \right) u_\mu u_\nu - \frac{P}{c^2} \, g_{\mu\nu}$$

where $u_\mu = g_{\mu\nu} u^\nu$ is the proper velocity. We then have

$$T_{\mu\nu} = \begin{Bmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{P}{c^2} e^{g+f} & 0 & 0 \\ 0 & 0 & \frac{P}{c^2} r^2 e^{g+f} & 0 \\ 0 & 0 & 0 & \frac{P}{c^2} r^2 \sin^2\theta \, e^{g+f} \end{Bmatrix}$$

Note that all off-diagonal velocity terms have been set to zero, since *co-moving* coordinates are assumed (an explanation of co-moving coordinates can be found in any cosmology text).

The Einstein field equations are of the form

$$R_{\mu\nu} - \frac{1}{2} \, g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} \, T_{\mu\nu}$$

To determine e^g and e^f and the FLRW metric, we will use the three field equations

$$R_{00} - \frac{1}{2}\, g_{00} R + \Lambda g_{00} = -\frac{8\pi G}{c^4}\, T_{00} \tag{1.1}$$

$$R_{11} - \frac{1}{2}\, g_{11} R + \Lambda g_{11} = -\frac{8\pi G}{c^4}\, T_{11} \tag{1.2}$$

$$R_{22} - \frac{1}{2}\, g_{22} R + \Lambda g_{22} = -\frac{8\pi G}{c^4}\, T_{22} \tag{1.3}$$

where the Ricci tensor is expressed as usual by

$$R_{\mu\nu} = \partial_\nu \Gamma^\alpha_{\mu\alpha} - \partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\lambda}\,\Gamma^\lambda_{\nu\alpha} - \Gamma^\alpha_{\mu\nu}\,\Gamma^\lambda_{\alpha\lambda}$$

The Christoffel symbols needed for the four Ricci tensors R_{00}, R_{11}, R_{22} and R_{33} and the Ricci scalar R are summarized in Adler et al. Those quentities are

$$R_{00} = \frac{3}{2}\, g'' + \frac{3}{4}\, g'^2$$

$$R_{11} = f'' + \frac{1}{r}\, f' - e^{g+f}\left(\frac{1}{2}\, g'' + \frac{3}{4}\, g'^2\right)$$

$$R_{22} = r^2\left(\frac{1}{2}\, f'' + \frac{1}{4}\, f'^2 + \frac{3}{2r} f'\right) - r^2 e^{g+f}\left(\frac{1}{2}\, g'' + \frac{3}{4}\, g'^2\right)$$

$$R_{33} = \sin^2\theta R_{22}$$

$$R = 3\left(g'' + g'^2\right) - 2\, e^{-(g+f)}\left(f'' + \frac{1}{4}\, f'^2 + \frac{2}{r}\, f'\right) \tag{1.4}$$

The primes denote ordinary differentiation with respect to their arguments, so that

$$g' = \frac{1}{c}\,\frac{dg}{dt}, \quad g'' = \frac{1}{c^2}\,\frac{d^2 g}{dt} \quad \text{and} \quad f' = \frac{df}{dr}, \quad f'' = \frac{d^2 f}{dr^2}$$

We will require only the first three field equations, which are

$$\frac{3}{2}\, g'' + \frac{3}{4}\, g'^2 - \frac{1}{2} R + \Lambda = K\rho \tag{1.5}$$

$$f'' + \frac{f'}{r} - e^{g+f}\left(\frac{1}{2}\, g'' + \frac{3}{4}\, g'^2\right) + \frac{1}{2}\, e^{g+f} R - e^{g+f}\Lambda = K e^{g+f}\,\frac{P}{c^2} \tag{1.6}$$

$$r^2\left(\frac{1}{2} + \frac{1}{4}\, f'^2 + \frac{3}{2r}\, f'\right) - r^2 e^{g+f}\left(\frac{1}{2}\, g'' + \frac{3}{4}\, g'^2\right) + \frac{1}{2}\, r^2 e^{g+f} R - r^2 e^{g+f}\Lambda = K r^2 e^{g+f}\,\frac{P}{c^2} \tag{1.7}$$

where $K = -8\pi G/c^4$. Note that (1.6) and (1.7) are identical except for the f terms, which allows for the general solution of e^f . Dividing out the r^2 in (1.7) and comparing, we see that

$$f'' - \frac{1}{2}\, f'^2 - \frac{1}{r}\, f' = 0$$

The solution to this differential equation is

$$e^f = \frac{1}{(1 + ar^2)^2} \tag{1.8}$$

where a is a constant to be determined. We can now use this identity to solve for e^g from (1.4), since R is assumed to be a constant. After calculating f' and f'' from (1.8), the most general solution to (1.4) is

$$e^g = a_1 e^{2\beta ct} + a_2 e^{-2\beta ct} + \frac{24a}{R} \tag{1.9}$$

where a_1, a_2 are constants and

$$\beta = \sqrt{\frac{R}{12}} \tag{1.10}$$

(Note that the dimensionality of R is length^{-2}, so that the argument of the exponential term is dimensionless, as required.) At this point it is convenient to set

$$a_1 = a_2 = \frac{1}{4} \quad \text{and} \quad a = \frac{1}{4}\beta^2$$

so that e^g and e^f reduce to

$$e^g = \cosh^2(\beta ct), \quad e^f = \frac{1}{(1+\frac{1}{4}\beta^2 r^2)^2} \tag{1.11}$$

which incidentally conform to the usual de Sitter solution for empty space. The FLRW metric is therefore

$$ds^2 = c^2 dt^2 - \cosh^2(\beta ct)\, \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}{(1+\frac{1}{4}\beta^2 r^2)^2} \tag{1.12}$$

An alternative (and perhaps more familiar) expression for this metric can be obtained using the coordinate transformation

$$u = \frac{r}{1+\frac{1}{4}\beta^2 r^2}$$

which results in

$$ds^2 = c^2 dt^2 - \cosh^2(\beta ct)\left[\frac{du^2}{1-\beta^2 u^2} + u^2 d\theta^2 + u^2 \sin^2\theta d\phi^2\right] \tag{1.13}$$

2. Inclusion of Density and Pressure

The FLRW metric (1.13) is identical to the traditional metric for empty de Sitter space, but we now need to examine our formalism with respect to the energy-momentum terms ρ and P. Having determined e^g and e^f, we now require only the two fields equations (1.5) and (1.6), which we reiterate here:

$$\frac{3}{2}g'' + \frac{3}{4}g'^2 - \frac{1}{2}R + \Lambda = K\rho \tag{2.1}$$

$$f'' + \frac{f'}{r} - e^{g+f}\left(\frac{1}{2}g'' + \frac{3}{4}g'^2\right) + \frac{1}{2}e^{g+f}R - e^{g+f}\Lambda = Ke^{g+f}\frac{P}{c^2} \tag{2.2}$$

At this point it is convenient to replace our definition of e^g with the traditional cosmological scalar expansion factor S^2, or

$$e^{g(ct)} = [S(ct)]^2$$

so that

$$g' = \frac{2S'}{S}, \quad g'' = \frac{2S''}{S} - \frac{2S'^2}{S^2}$$

After eliminating the f terms using (1.11), we insert these expressions into (2.1) and (2.2), arriving at

$$\frac{3S''}{S} = K\rho + \frac{1}{2}R - \Lambda \tag{2.3}$$

and

$$\frac{S''}{S} + \frac{2S'^2}{S^2} = -K\frac{P}{c^2} + \frac{1}{2}R - \Lambda + \frac{2\beta^2}{S^2} \tag{2.4}$$

Now, since we can take the square root of $S^2 = \cosh^2\beta ct$, we find that there are two possibilities for the scale factor,

$$S = \cosh(\beta ct) \quad \text{and} \quad S = -\cosh(\beta ct) \tag{2.5}$$

which surprisingly yield two distinct results. Let us consider the negative root first. We then have

$$S' = -\beta \sinh(\beta c t), \quad S'' = -\beta^2 \cosh(\beta c t) = S\beta^2 \tag{2.6}$$

Inserting these quantities into (2.3) and (2.4), it is then a simple matter to show that

$$K\left(\rho - \frac{3P}{c^2}\right) = 4\Lambda - R \tag{2.7}$$

In a pure de Sitter space devoid of all matter and energy we have $R = 4\Lambda$, and we would like to preserve this identification for the present situation. We thus have the cosmological equation of state

$$\frac{P}{c^2} = \frac{1}{3}\rho \tag{2.8}$$

which applies to a pure radiation field of relativisitic massless (or nearly massless) particles, like photons and neutrinos. For the postive root $S = \cosh(\beta c t)$, we have instead

$$S' = \beta \sinh(\beta c t) \quad \text{and} \quad S'' = \beta^2 \cosh(\beta c t) = S\beta^2 \tag{2.9}$$

Inserting these quantities into (2.3) and (2.4) as before, we have the even simpler result

$$\frac{P}{c^2} = -\rho \tag{2.10}$$

This describes a *negative pressure* in the presence of a field of mass-energy having a non-vanishing proper density. This is the case for dark energy, which is assumed to be the cause of our universe undergoing accelerated expansion due to the repulsive effect of dark energy.

3. Conclusions and Comments

The cosmological equation of state for a perfect fluid of non-interacting particles is given by

$$\frac{P}{c^2} = \omega\rho$$

where ω is the usual dimensionless equation of state parameter. In the present case with a constant curvature scalar R we have two possibilities, $\omega = 1/3$ and $\omega = -1$, corresponding to a very early and very late FLRW universe, respectively. These results suggest that a constant Ricci scalar R is valid for both cases, although the formalism says nothing about its value in a matter-dominated universe, where R would likely vary with time.

The formalism also says nothing about any detailed behavior of the scale factor S given by (2.5), which eliminates the derivation of expressions relating to the traditional Friedmann equations for a perfectly isotropic, homogeneous universe. Indeed, the behavior of S depends only on the sign of the Ricci scalar which, if positive, results in an exponentially expanding universe. If R is zero, the FLRW metric degenerates into flat Minkowki spacetime, as expected, while for $R < 0$ the scale factor becomes sinusoidal, describing a universe that expands and contracts periodically with time.

References

1. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, McGraw-Hill, 2nd Edition, 1975. Chapter 12 provides a detailed derivation and summary of the Christoffel symbols required for the construction of the Ricci tensors $R_{\mu\nu}$ and the Ricci scalar R presented in Section 1.

2. G. 't Hooft, *Introduction to General Relativity*, 2002. This is a short paper available for download at

   http://srv2.fis.puc.cl/ mbanados/Cursos/RelatividadGeneral/tHooft-Notes.pdf

   't Hooft uses a different approach in the derivation of the FLRW metric based on a maximally-symmetric 3-space. His discussion is not limited to de Sitter spacetime and includes a review of the usual Friedmann equations dealing with Hubble's velocity-distance relation and related issues.

3. C. Pearson, *Fundamentals of Cosmology*, 2003. There are many excellent texts on physical cosmology available, but this entertaining and informative eight-part presentation provides all the basic information needed for an elementary understanding of the subject. The author uses $R(t)$ as the scale factor, an unfortunate choice considering its possible confusion with the curvature scalar R. The text is out of print, but can be downloaded as a slide presentation from

   http://www.weylmann.com/Pearson.zip

4. W. Straub, *The Friedmann-Lemaître-Robertson-Walker Metric in a de Sitter Universe*, December 21, 2019. Related supplementary information is available from this paper, which can be downloaded from

   http://www.weylmann.com/flrw.pdf