Geodesic Curve of a Gravitational Plane Wave Pulse and Curve of Particle

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Abstract

We consider a system of a gravitational wave coming from infinity that collides with a mass M. The metric of the system approaches the metric of a gravitational plane wave pulse as the mass of M goes to zero. The metric of the gravitational plane wave pulse having a specific form. We show there is a limiting curve of M, as mass and size of M go to zero, that is not a geodesic curve. We show conservation of energy-momentum does not hold and there is no solution to the Einstein field equations for this system.

1 Gravitational plane wave pulse metric

Define $u = t - x$ and let the metric $g_{\mu\nu}(u)$ be [1]

$$
\begin{aligned}
g_{00}(u) &= -1 && (1)\\
g_{11}(u) &= 1 && (2)\\
g_{22}(u) &= [L(u)]^2 e^{2\beta(u)} && (3)\\
g_{33}(u) &= [L(u)]^2 e^{-2\beta(u)} && (4)\\
g_{01}(u) &= g_{02}(u) = g_{03}(u) = g_{12}(u) = g_{13}(u) = g_{23}(u) = 0 && (5)
\end{aligned}
$$

having $g_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$ and

$$
\frac{d^2L}{du^2}(u) + \left[\frac{d\beta}{du}(u)\right]^2 L(u) = 0 \qquad (6)
$$

This metric will satisfy $R_{\mu\nu} = 0$. It is the metric of a gravitational plane wave pulse. Let $L(0) = 1$ and $\beta \neq 0$. We then have by (6) that $L(u)$ will decrease and become zero at some point $u_0 > 0$. Consequently $g_{22}(u) > 0$ for $u < u_0$.

2 Proper Lorentz transformation

Consider a coordinate transformation from t, x, y, z to t', x', y', z' coordinates that is a composition of a rotation by θ about the z axis followed by a boost by $2\cos\theta/(1+\cos^2\theta)$ in the x direction followed by a rotation by $\theta + \pi$ about the z axis. For θ/π not an integer this is a proper Lorentz transformation such that

$$
\begin{aligned}
t &= t'(1 + 2\cot^2\theta) - 2x'\cot^2\theta + 2y'\cot\theta && (7)\\
x &= 2t'\cot^2\theta + x'(1 - 2\cot^2\theta) + 2y'\cot\theta && (8)\\
y &= 2t'\cot\theta - 2x'\cot\theta + y' && (9)\\
z &= z' && (10)
\end{aligned}
$$

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By (7) and (8) we have $t - x = t' - x'$. For (7)-(10) define the metric $g'_{\mu\nu}(u)$ by

$$g'_{\mu\nu}(u) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(u) \tag{11}$$

hence for the metric (1)-(5) we get

$$\begin{aligned}
g'_{00}(u) &= -1 - 4[1 - g_{22}(u)]\cot^2\theta && (12)\\
g'_{01}(u) &= 4[1 - g_{22}(u)]\cot^2\theta && (13)\\
g'_{11}(u) &= 1 - 4[1 - g_{22}(u)]\cot^2\theta && (14)\\
g'_{02}(u) &= -g'_{12}(u) = -2[1 - g_{22}(u)]\cot\theta && (15)\\
g'_{22}(u) &= g_{22}(u) && (16)\\
g'_{33}(u) &= g_{33}(u) \qquad g'_{03}(u) = g'_{13}(u) = g'_{23}(u) = 0 && (17)
\end{aligned}$$

The metric $g'_{\mu\nu}(u)$ satisfying $R_{\mu\nu} = 0$ and $g'_{\mu\nu}(u) = \eta_{\mu\nu}$ for $u < 0$ is then also the metric of a gravitational plane wave pulse.

3 Geodesic curve

The curve

$$\begin{aligned}
t(\lambda) &= (1 + 2\cot^2\theta)\lambda - 2\cot^2\theta \int_0^\lambda \frac{dw}{g_{22}(w)} && (18)\\
x(\lambda) &= 2\cot^2\theta\lambda - 2\cot^2\theta \int_0^\lambda \frac{dw}{g_{22}(w)} && (19)\\
y(\lambda) &= -2\cot\theta\lambda + 2\cot\theta \int_0^\lambda \frac{dw}{g_{22}(w)} && (20)\\
z(\lambda) &= 0 && (21)
\end{aligned}$$

satisfies the geodesic equation for the metric $g'_{\mu\nu}(u)$ and so is a geodesic curve. We have for $\lambda < 0$, since $g_{22}(u) = 1$ for $u < 0$, that $x(\lambda) = y(\lambda) = z(\lambda) = 0$. Choose θ so that $\cot\theta \neq 0$. We then have by (18), since the integral goes to positive infinity as $\lambda \to u_0$, that $t(\lambda) \to -\infty$ as $\lambda \to u_0$.

4 Geodesic curve and curve of particle

Consider a system of a gravitational wave pulse coming from negative x infinity that collides at $t = 0$ at the origin with a mass M initially at rest. Let $\tilde{g}_{\mu\nu}(t, x, y, z)$ be the metric of the system. Require $\tilde{g}_{\mu\nu}(t, x, y, z) \to g'_{\mu\nu}(t - x)$ as the mass of M goes to zero.

For $t < 0$ as t gets closer and closer to zero the wave front of the gravitational wave pulse gets closer and closer to M. For $t < 0$ the wave front makes no contact with M hence for $t < 0$ there is no mass inside the wave.

Let M have small mass and size and assume the curve of M is approximately the geodesic curve (18)-(21). For $\lambda > 0$ points of the geodesic curve are inside the wave. Now $t(\lambda) \to -\infty$ as $\lambda \to u_0 > 0$. We can then conclude there is a point with negative t inside the wave where there is a mass. This contradicts the previous paragraph.

Since the curve of M is not approximately a geodesic curve we have the limiting curve, as mass and size go to zero, of M cannot be a geodesic curve.

5 Conservation of energy-momentum and existence of solution

At a point p choose coordinates t'', x'', y'', z'' so that the metric $\tilde{g}''_{\mu\nu}$ is the Minkowski metric and the first order partial derivatives of the metric are zero.

We have by conservation of energy and momentum that $T''^{\mu\alpha}_{;\alpha} = 0$ or

$$\frac{\partial T''^{\mu\alpha}}{\partial x''^{\alpha}} + \Gamma''^{\mu}_{\alpha\beta} T''^{\alpha\beta} + \Gamma''^{\alpha}_{\alpha\beta} T''^{\mu\beta} = 0 \tag{22}$$

where $T''^{\mu\nu}$ is the energy-momentum tensor of M in t'', x'', y'', z'' coordinates and $\Gamma''^{\mu}_{\alpha\beta}$ is calculated using $\tilde{g}''_{\mu\nu}$. We then have $\Gamma''^{\mu}_{\alpha\beta}$ are zero at p so by (22) we have at p that $\partial''_{\alpha} T''^{\mu\alpha} = 0$. Since the curve M is not approximately a geodesic there is a p on the curve of M so that M experiences an acceleration in the t'', x'', y'', z'' coordinate system and hence $\partial''_{\alpha} T''^{\mu\alpha}$ is not zero at p. Consequently (22) does not hold. Also there is no solution to the Einstein field equations for this system since a solution would have $T''^{\mu\alpha}_{;\alpha} = 0$.

6 Conclusion

There is a limiting curve of a mass, that a gravitational wave pulse collides with, that is not a geodesic. We showed conservation of energy-momentum does not hold and there does not exist a solution to the Eistein field equations for this system.

References

[1] C. M. Misner, K.S. Thorne, and J. A. Wheeler, Gravitation p. 957 (W.H. Freeman, San Francisco, CA, 1973)

[2] K. De Paepe, Physics Essays, June 2018