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Assuming *ABC* Conjecture is True Implies Beal Conjecture is True

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Abstract In this paper, we assume that the *ABC* conjecture is true, then we give a proof that Beal conjecture is true. We consider that Beal conjecture is false then we arrive to a contradiction. We deduce that the Beal conjecture is true.

Keywords *ABC* Conjecture · *ABC* Theorem · Beal conjecture · Diophantine equations.
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To the memory of Jean Bourgain (1954-2018) for his mathematical work notably in the field of Number Theory

1 *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

A paper about the proof of the *ABC* conjecture, that is true [1], was submitted recently (December 2018) to the journal Research In Number Theory. We have obtained the following theorem:

Theorem 1 *(**ABC Theorem**): For each $\varepsilon > 0$, there exists $K(\varepsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\varepsilon).rad(abc)^{1+\varepsilon} \tag{3}$$

where K is a constant depending only of ε equal to $\dfrac{2}{\varepsilon^2}$.

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In 1997, Andrew Beal [2] announced the following conjecture:
Conjecture 1: (Beal Conjecture)
Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:

$$A^m + B^n = C^l \tag{4}$$

then $A, B,$ and C have a common factor.

2 *Methodology of the proof*

We note :

$$\text{A: Beal Conjecture} \tag{5}$$
$$\text{B: } ABC \text{ Theorem} \tag{6}$$

and we use the following property (the contrapositive law, [3]):

$$\boxed{A(False) \Longrightarrow B(False)} \Longleftrightarrow \boxed{B(True) \Longrightarrow A(True)} \tag{7}$$

From the right equivalent expression in the box above, we obtain that B (*ABC* Theorem) which is true implies A (Beal Conjecture) is true.

3 *Proof of the conjecture* (1)

We suppose that Beal conjecture is false, then it exists A, B, C positive **coprime** integers and m, n, l positive integers > 2 such:

$$A^m + B^n = C^l \tag{8}$$

the integers A, B, C, m, n, l are supposed large integers. We consider in the following that $A^m > B^n$. Now, we use the *ABC* theorem for equation (8) because A^m, B^n, C^m are relatively coprime. We obtain :

$$C^l < K(\varepsilon) rad(A^m B^n C^l)^{1+\varepsilon} \Longrightarrow C^l < K(\varepsilon) \left(rad(A).rad(B).rad(C)\right)^{1+\varepsilon} \tag{9}$$

As $rad(A) \leq A, rad(B) \leq B$ and $rad(C) \leq C$, the last equation becomes:

$$C^l < \frac{2}{\varepsilon^2}(A.B.C)^{1+\varepsilon} \tag{10}$$

But $rad(A) \leq A < C^{\frac{l}{m}}$, $rad(B) \leq B < C^{\frac{l}{n}}$, then we write (10) as :

$$\frac{\varepsilon^2}{2} < C^{\left(1+\frac{l}{m}+\frac{l}{n}\right).(1+\varepsilon)-l} \tag{11}$$

3.1 *Case $m > l$ and $n > l$*

In this case, $\left(1+\dfrac{l}{m}+\dfrac{l}{n}\right).(1+\varepsilon)-l \approx 3-l+3\varepsilon$. We take $\varepsilon = 1$. As $6 \ll l \Longrightarrow \dfrac{1}{C^{l-6}} \ll 0.5$, then the contradiction.

3.2 *Case $m < l$ and $n < l$*

In this case, if $C > A \Rightarrow C^m > A^m > B^n \Rightarrow C^m > B^n \Rightarrow C^m > C^l - A^m \Rightarrow A^m > C^l - C^m \Rightarrow A^m > C^m(C^{l-m} - 1)$. As $l > m \Rightarrow C^{l-m} - 1 > 1$, then $A^m > C^m \Longrightarrow A > C$ that is a contradiction with $C > A$. Hence $C < A$. We rewrite equation (10):

$$C^l < K(\varepsilon) rad(A^m B^n C^l)^{1+\varepsilon} \Longrightarrow C^l < K(\varepsilon)(A.B.C)^{1+\varepsilon}$$
$$\Rightarrow A^m < C^l < K(\varepsilon)\left(A.A^{\frac{m}{n}}.A\right)^{1+\varepsilon} \tag{12}$$

Then:

$$A^m < \frac{2}{\varepsilon^2}.A^{\left(2+\frac{m}{n}\right)(1+\varepsilon)} \tag{13}$$

3.2.1 Case $n > m$

If $n > m$, we have $\left(2+\dfrac{m}{n}\right)(1+\varepsilon) \approx 3+3\varepsilon$. We take $\varepsilon = 1$, as $6 \ll m \Longrightarrow \dfrac{1}{A^{m-6}} \ll 0.5$, then the contradiction.

3.2.2 Case $n < m$

We have:

$$C^l < K(\varepsilon)(A.B.C)^{1+\varepsilon} \tag{14}$$

As $A^m < C^l, C < A$ and $B^n < A^m \Longrightarrow B < A^{m/n}$, the last equation becomes:

$$\frac{\varepsilon^2}{2} < A^{(2+m/n)(1+\varepsilon)-m} \tag{15}$$

We choose $\varepsilon = \dfrac{1}{m}$, we obtain :

$$\frac{1}{2m^2} < A^{2-m+\frac{2}{m}+\frac{1}{n}} \Longrightarrow \frac{1}{2m^2} < A^{3-m} \tag{16}$$

But $3 \ll m$ and $1 \ll A \Longrightarrow \dfrac{1}{2m^2} > A^{3-m}$, then the contradiction.

3.3 Case $m < l$ and $n > l$

If $C < A$, as $l < n \Rightarrow C^l < A^n \Rightarrow 0 < A^m < A^n - B^n$ then $A > B$. As $C^n > C^l > B^n \Rightarrow C^n > B^n \Rightarrow C > B$. So we obtain :

$$\boxed{B < C < A} \tag{17}$$

Then the equation (10) becomes:

$$C^l\frac{\varepsilon^2}{2} < (A.B.C)^{1+\varepsilon} \Longrightarrow C^l\frac{\varepsilon^2}{2} < (A.A^{l/n}.A)^{1+\varepsilon} \Rightarrow C^l\frac{\varepsilon^2}{2} < A^{(2+l/n)(1+\varepsilon)} \tag{18}$$

As $A^m < C^l$, we arrive to:

$$\frac{\varepsilon^2}{2} < A^{3-3m+3\varepsilon} \tag{19}$$

We take $\varepsilon = 1/3 \Longrightarrow A^{3m-4} < 18$, then the contradiction because $18 \ll A^{3m-4}$.

If $A < C \Rightarrow A^l < C^l$ but $B^n < A^m \Rightarrow A^l < 2A^m \Rightarrow A^l < A^{m+1} \Rightarrow l < m+1$, as $m < l \Rightarrow m+1 \leq l < m+1$ that is a contradiction, then $C < A$ and this case is studied above.

3.4 Case $m > l$ and $n < l$

We have $n < l < m$. As $A^m < C^l \Rightarrow A < C^{\frac{l}{m}} < C \Rightarrow A < C$. The equation (10) becomes:

$$C^l < \frac{2}{\varepsilon^2}\left(C^{l/m}.C^{l/n}.C\right)^{1+\varepsilon} \tag{20}$$

We take $\varepsilon = 0.1$, we obtain:

$$0.005 < C^{2.2+1.1\frac{l}{n}-l} \approx C^{3+\frac{l}{n}-l} \tag{21}$$

But as $3 \ll l \Longrightarrow l > 3 + \frac{l}{n}$, then the contradiction.

All the cases give contradiction, then *ABC* theorem is false. We deduce from :

$$\boxed{\text{Beal Conjecture (False)} \Rightarrow \text{ABC Theorem (False)}} \Leftrightarrow \boxed{\text{ABC Theorem (True)} \Rightarrow \text{Beal Conjecture (True)}}$$

that Beal Conjecture is true.

The proof of the Beal conjecture is achieved[1].

4 *Conclusion*

From the *ABC* theorem, we have given a proof that the *ABC* conjecture is true. We can announce the theorem:

Theorem 2 ***(Abdelmajid Ben Hadj Salem, Andrew Beal, 2019):*** *Let $A,B,C,m,n,$ and l be positive integers with $m,n,l > 2$. If:*

$$A^m + B^n = C^l \tag{22}$$

then $A,B,$ and C have a common factor.

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[1] A paper giving another proof of Beal conjecture is under reviewing [4]