A Test of the Superposition Principle in Intense Laser Beams

Alexander Unzicker
Pestalozzi-Gymnasium München
e-mail: aunzicker@web.de

January 6, 2019

Abstract

A test of a nonlinear effect hitherto unknown in classical electrodynamics is proposed. For the possible nonlinearity to be observed, a high-intensity standing wave with circular polarization in a resonator is required. If the effect exists, an electric voltage should be induced between the mirros and an electric current can be measured. Motivation, quantitative expectations and the design of the experiment are discussed.

1 Introduction

In current fundamental research, high-risk experiments are relatively rare, while low-risk projects with largely expected results dominate scientific practise. The present proposal with its suggestion is high-risk in the sense that such an effect is unexpected by current theories, though no elementary priciples such as symmetry forbid its existence.

Such an effect may be considered unlikely, however it would be quite consequenctial for our understanding of electrodynamics. Therefore, I shall restrict to a short outline of the experiment's motivation.

Though not everyone seems to be aware of it, classical electrodynamics is inconsistent. This is a direct consequence of the infinite self-energy of the electric field of an electron (see textbooks such as Feynman's [1], chap. 28, or Landau's [2], chap. 75). Therefore, a generally valid formula that describes how accelerated charges emit radiation is still unknown - a problem which is *not* resolved by quantum electrodynamics either [1]. Thus there are still fundamental challenges in our understanding of the interaction of electric charges with light.

That said, unusual ideas about electrodynamics may be worth testing. It is widely known that the ether theories of the 19th century were in agreement with Maxwells's equations [3], though the concept of the ether was considered superfluous after Einstein's formulation of special relativity. It is much less known however, that the idea of an elastic solid does not contracdict relativity. Instead, relativistic effects such as length contraction are a consequence of the respective wave equations [4].

However, I shall not discuss ether concepts at length here, since the suggested effect does not really depend on that idea. Rather it is a consequence of a class of nonlinearities compatible with a variety of theoretical models. It is however appropriate to give the Scottish physicist James MacCullagh due credit for his idea of the electric field being related to a rotation of volume elements. Indeed, his theory from 1839 is identical with Maxwell's equations for empty space [3]. While MacCullagh considered the linear case, I shall analyse the strong fields nonlinearities that should appear.

2 Theoretical background

This section gives a brief account of how the above hypothesis leads to quantitative predictions. The reader interested in experimental details may skip most of it and jump to the summary paragraph 2.3 at the end of the section.

2.1 Mathematical properties of rotations in 3-space

The group $SO(3)$. Rotations in threedimensional space are described by the group $SO(3)$, usually represented by 3x3 matrices which can be computed from subsequent rotations by means of Euler angles [5].[1] Each element of $SO(3)$ can also be visualized by a vector, whereby the direction indicates the axis of rotation, while length denotes the amount of rotation, the total angle. For any given axis however, the angles π and $-\pi$ are identical, making the vector representation invalid for such large angles.

Hence, elements of $SO(3)$ are not vectors, but the superposition principle for the electric field requires precisely that. Only if the electric field is truly represented by a vector, the addition of components pointing in different directions makes sense. However, if the electric field is a rotation, it would just appear similar to a vector and such an addition would not make sense any more. In fact, rotations in threedimensional space do not commute, and the correct operation to describe to subsequent rotations is matrix multiplication, not vector addition. Only in the case of small amplitudes the vector addition is a valid approximation - the weak field case of electrodynamics. Because strong fields are rare and no particular interest has been given to such a specific case, the abovementioned nonlinearity could have gone unnoticed so far.

However, since the noncommutativity of thredimensional rotations, expressed by matrix multiplications, does lead to computable results, the consequences of a deviation from the superpostion principle for strong electric fields can be predicted in principle.

Let's give a concrete example. While for real numbers the multiplication $3 \cdot 5 \cdot \frac{1}{3}\frac{1}{5}$ trivially commutes, this is not the case for matrices representing rotations in 3-dimensional space. If Y and Z denote clockwise rotations around the respective axis and Y^{-1}, Z^{-1} the corresponding counterclockwise rotation (inverse matrix), then $YZY^{-1}Z^{-1}$ creates X, a slight rotation around the x-axis. Thus alternating rotations around to two different axes would lead to a small rotation around the perpendicular axis. If one follows the hypothesis of the electric field being a rotation of volume elements, such a nonlinearity must be implemented in an experiment.

Identification with light. For a test of the above hypothesis, a physical equivalent of alternating rotations $YZY^{-1}Z^{-1}$ must be found. If one considers a standing light wave of circular polarization extending in x-direction, in every non-knot region this is the case, since the electrical field successively points to the corresponding directions. The only difference is that in the light wave, there is a continuous change of the field direction while we have considered a discrete sequence above.

To recapitulate, a circularly polarized standing wave may be visualized in two ways: as a superposition of two standing waves linearly polarized in y- and z- direction, with an interjacent phase shift of $\frac{\lambda}{4}$. Or as a superposition of circularly polarized waves propagating in positive and negative x-direction. Mind that the $\frac{\lambda}{2}$ - jump at the reflecting mirror causes the necessary condition - the angular momentum contained in the light wave is just reflected, there is no transfer to the mirror. Thus, the circularly polarized standing wave carries angular momentum along the x-axis, a necessary asymmetry for the following considerations.

If one considers now the electric field being a rotation, the four subsequent states of the light waves $t = 0$, $t = \frac{T}{4}$,$t = \frac{T}{2}$ and $t = \frac{3T}{4}$ in which $\vec{E}$ is parallel to the y- and z-axis would represent the above sequence of rotations $YZY^{-1}Z^{-1}$ from which one may expect that a z-rotation is generated. However, for a quantitative analysis, the steady change of the electric field vector has to be modelled by a series

[1] There are very interesting topological peculiarities of $SO(3)$ and its double cover $SU(2)$, also called 'spin group' that cannot be discussed here.

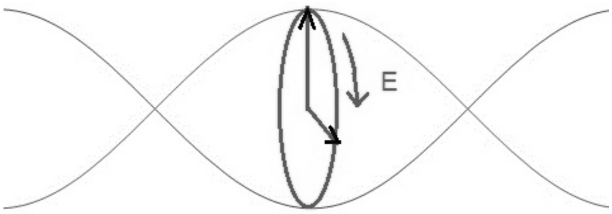

Figure 1: Standing wave of circularly polarized light propagating in x-direction. Between the knots, the vector of the electrical field $\vec{E}$ rotates in a plane perpendicular to propagation

real rotations whose *axes* lie in y-z-plane. This is similar to a numerical integration, the summing up however to be replaced by a matrix multiplication.

Implementation with computer algebra. The sequence of matrix multiplications can easily done by a computer algebra system, the first factor being

$$\begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \tag{1}$$

which represents a rotation around the y-axis. θ determines the strength of the electric field and will be a variable in the program. Only two other parameters occur, the number of oscillation periods k and the number of steps per oscillation period n. The phase shift between y- and z-direction must be $\varphi = \frac{\pi}{2}$ for circular polarization.[2] Mind that φ denotes the conventional phase of the light wave, while θ describes the rotation of volume elements, if the hyothesis to be tested is valid. At the very end it is of interest what rotation around the x-axis (if any) is induced by the described sequence. The total angle θ_x can be computed from the resulting matrix M by

$$\cos\theta_x = \frac{1}{2}(tr\ M - 1), \tag{2}$$

a formula known since Kelvin and Tait (1896). The complete Mathematica code can be found in the appendix.

2.2 Physical interpretation of the predictions

Model results. Running the matrix multiplication with a given amplitude θ yields a θ_x that converges for large n ($n = 10.000$ is easily done within a minute). Of course, besides the absolute value one has to determine how θ_x depends on θ. Tab. 1 displays some results.

While for $\theta = 2$ is close to the maximum value π, numerical artefacts appear to occur for $\theta_x < 10^{-6}$. It is however clear that θ_x depends quadratically on θ, which is reflected by the steepness $\frac{1}{2}$ of the Log-Log-Plot Fig. 2, which is just another form of displaying the results. Keeping in mind that the analysis hitherto describes one oscillation period and may easily be extended to a longer duration. Fig. 3 shows the accumulated effect for five oscillation periods, for vizualisation puropses with a large angle $\theta = 1$. Note that in the middle of a oscillation period the rotation around the x-axis may be larger than after completing the period (minima in fig. 3).

Since the above results refer to *one* oscillation period, the obtained angle θ_x therefore will represent an *increase* of the electric field during the corresponding time $T = \frac{1}{f}$. The identification with the SI units $\frac{V}{ms}$ will be done in the next paragraph.

[2]By taking other values, the program may analyse imperfect situations as well.

θ	θ_x
2	0.3106
1	0.07902
0.5	0.01986
0.2	0.00318
0.1	0.0007957
0.01	0.0000080
0.001	0.00000105

Figure 2: Numerical results for the rotation around the x-axis as a consequence of steadily alternating rotations around the y- and z-axis.

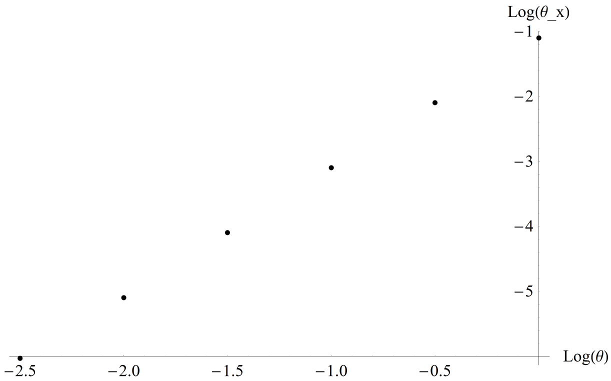

Figure 3: Resulting rotation angle around the x-axis depending on the imput rotation angle, in double logarithmic plot. Obviously, the effect grows quadratically with the input θ.

Identification with electrical units. The hypothesis that electric fields being rotations implies that there is a maximum value of the electric field in nature corresponding to π, but ine still has to express that in the SI unit $\frac{V}{m}$. The transformation coefficient is not known exactly. However we may assume that such strong electric fields occur in the vicinity of an eletric charge, and we shall further assume that the electron mass is contained in the potential energy of its fields - an idea put forward already by Hendrik Antoon Lorentz around 1900. It leads to the so-called classical electron radius r_c obtained by the condition

$$\int_{r_c}^{\infty} \frac{1}{2}\epsilon_0 \vec{E}(\vec{r})^2 dV = m_e c^2. \tag{3}$$

since the cutoff $r_c = 1.4 \cdot 10^{-15} m$ describes the region where the known laws of electrodynamics must break down, $E(r_c)$ is a reasonable measure of the maximum possible value which for obvious mathematical reasons corresponds to π. Easy evaluation leads to

$$E_c = \frac{e}{4\pi\epsilon_0 r_c^2} = 7 \cdot 10^{20} \frac{V}{m}. \tag{4}$$

However, it can also be argued that classical approximations break down at the de Broglie- wavelength of the electron, $\lambda = \frac{h}{cm_e} \approx 2.42 \cdot 10^{-12}\ m$. This would lead to a much smaller value of $E_b = 2.4 \cdot 10^{14} \frac{V}{m}$. And yet another estimate is the so-called Schwinger limit in quantum electrodynamics, the electric field strenght where the spontaneous creation of electron-positron pairs becomes likely. The common estimate is $E_s \approx 10^{16} \frac{V}{m}$. Since there is a considerable difference between these values, the following analysis will be done for all of them. In terms of radiation density, E_c, E_s and E_b correspond to $(w_i = \frac{1}{2} c\epsilon_0 E^2)$

$$(w_c, w_s, w_b) = (7 \cdot 10^{38} \frac{W}{m^2}, 10^{29} \frac{W}{m^2}, 8 \cdot 10^{25} \frac{W}{m^2}), \tag{5}$$

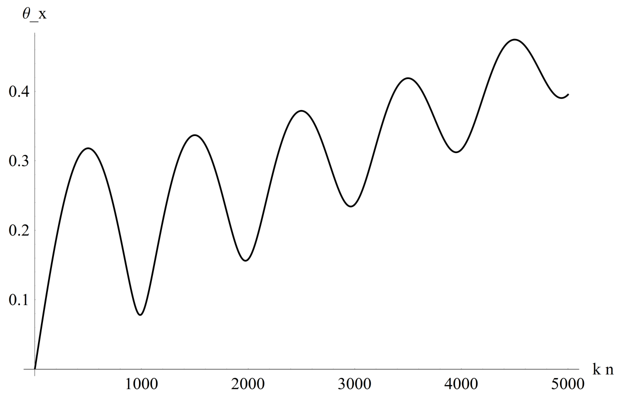

Figure 4: Total rotation around the x-axis after five oscillation periods

the last (de Broglie) case being the easiest one to detect, the ‘classical’ case the hardest one.

Electric effect. In our modeling assumptions, we considered the situation at the highest amplitude zone of the circularly polarized standing wave (exemplary circle in fig. 1), thus for the cumulated effect over a finite distance between the mirrors, one must reduce it by a factor computed by the Integral

$$\frac{1}{2\pi}\int_0^{2\pi}(\sin(k\ x))^2 dx = \frac{1}{2}. \qquad (k \in Z) \tag{6}$$

Thus, if we still think the electric field being a dimensionless quantity describing rotations, the increase *per oscillation period* will be

$$\theta_x = \frac{1}{2} 0.08\ \theta^2, \tag{7}$$

taking the above numerical result into account. Being quadratically with the field, it follows that the effect will increase *linearly* with light intensity. All this refers to the simplified case of planparallel mirrors. The more sophisticated beam profiles necessary due to the stability of the cavity should however not change the order of magnitude of the effect.

2.3 Summary of model predictions.

Voltage. Assuming that π corresponds to one of the maximum fields E_c, E_s, E_b, any pure angle may be translated into the physical unit $\frac{V}{m}$.

I will first give an example and then explain how the suspected effect depends on the various quantities. If the laser power $P = 4\ W$ is concentrated to a beam width of $q \approx\ mm^2$, the intensity w amounts to $10^4 \frac{W}{m^2}$ outside the cavity, which can be enhanced by a factor depending on the mirror reflectance. In a 1064 nm laser[3] with $f = \frac{c}{\lambda}$, the duration of one oscillation period is $T = \frac{1}{f} = 3.55$ fs. Assuming further a mirror distance $d = 10\ cm$ and a coherence lenght of the laser of about $L \approx 50\ m$ (that means that the electric field may at best accumulate for the corresponding coherence time $\frac{L}{c} \approx 167\ ns$), during that period one should be able to measure a voltage increase of $U_c = 40\ \mu V$, $U_s = 118\ V$ or $U_b = 3.3\ V$, depending on the previous assumptions on the electron radius. If the above quantities are changed, the dependence of the resulting voltage U is

$$U \sim \frac{PL\ d\ E}{q\lambda}, \tag{8}$$

[3]At 1064 nm it is much easier to obtain high-reflectances of metal mirrors, while at $\lambda = 532\ nm$ a reflectance of 0.96 can hardly be superated, see Wikipedia: Optical coating.

E being the power enhancement factor of the cavity. For a reflectance of 0.98, the enhancement in a 4-cavity (see below, sec. 3 and [6], p. 44) calculates as

$$E = (\frac{t}{1 - r_d^2 r_m^2})^2, \tag{9}$$

r_d and r_m being the reflection coefficients[4] of the dielectric and metallic mirrors, respectively. t is the transmission coefficient. With a reflectance of 0.98, $E = 12$ can be obtained (0.99 leads to $E = 25$).

Current. Since it is not quite evident how the voltage increase can be maintained under experimental conditions, it is useful to look at the predictions also in terms of electric current. According to Maxwell's equations, a increase of the electric field $\frac{d\vec{E}}{dt}$ corresponds to a current density $\frac{\vec{j}}{\epsilon_0}$. Contrary to the voltage increase, the total current would be independent from the mirror distance d and from the correlation length L of the beam. More importantly, the current is also independent from the beam cross section q, since $\vec{j}$ has to be integrated over q and compensates the loss of intensity. This eases the predictions because more complicated beam profiles then should have a marginal influence. The calculation yields $I_c = 17$ fA, $I_s = 1.4$ nA and $I_b = 49$ nA, again depending on the electron radius. The general dependence reduces to

$$I \sim \frac{PE}{\lambda}. \tag{10}$$

Thus the main experimental task is to increase the enhancement of the cavity E.

Resistance of the mirror coating. The current to measure has to pass the metal coating of the mirror, therefore it is useful to consider the coating's electrical resistance. With a beam width of $r = 0.5\ mm$, a mirror radius (position of the electric contact) $D = 12.5\ mm$, and a coating thickness $d = 100\ nm$, one obtains the electrical resistance Re by integration

$$Re = \int_r^R \frac{\sigma}{2\pi \rho d} = \frac{\sigma}{2\pi d} \ln \frac{R}{r} = 0.115\ \Omega, \tag{11}$$

$\sigma = 22.4 \cdot 10^{-9}\ \Omega m$ being the conductivity of gold. As electric contacts, indium rings may be pressed on the gold mirrors.

3 Experimental Setup

3.1 General design.

Laser cavity. Since intensity plays a crucial role, it is advantageous to trap the laser beam in a Fabry-Perot-like cavity (see fig. 5). To achieve the appropriate phase condition for amplification, dielectric mirrors are required. On the other hand, for the electric measurement metallic mirrors are necessary. The only feasible solution seems to be a four-mirror-cavity of the form fig. 5. However, to avoid any asymmetry for the electric measurement, the cavity should have an input beam from both sides (the initial beam splitting is not shown in fig. 5). A fine-tuning of both mirror positions is crucial, since the resonator length has to be a multiple of $\frac{\lambda}{2}$. The peculiarity of the setup is that the four mirrors are not, as usual used for a ring laser system but still contain a standing wave.

Stability and mirror curvature. The optical resonator must be in a stable region determined by the mirror radii.[5] For the purpose of the experiment, a Gaussian beam profile (e.g. [6]) is an excellent approximation. A stable region of the resonator can be obtained either with radii between

[4] Refering to the electric field, they are calculated by the square root of the reflectance.

[5] see, e.g. Wikipedia: Optical cavity.

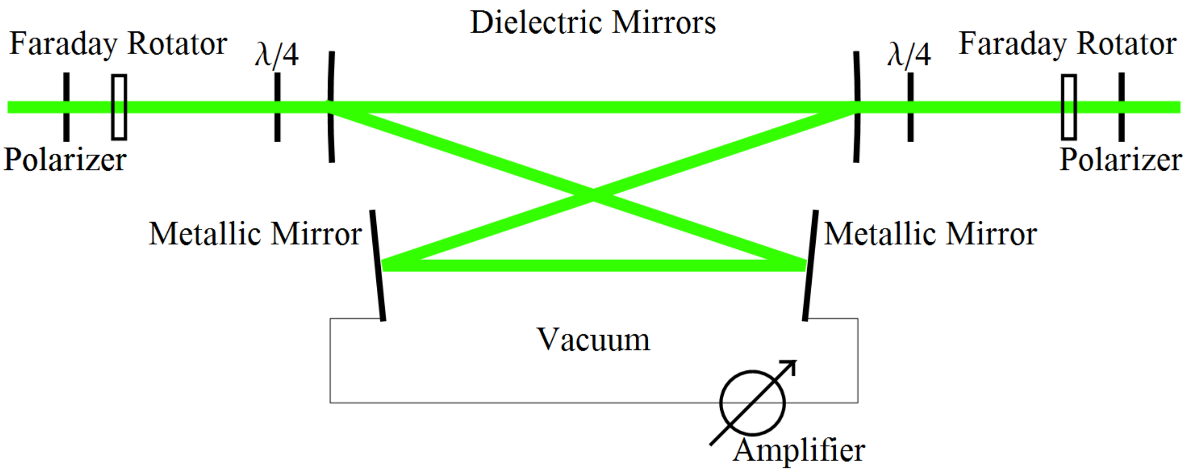

Figure 5: Schematic picture of the experiment.

the confocal ($R_1 = R_2 = L$) and the planparallel case or between the confocal and the concentric case ($R_1 = R_2 = L/2$). It is not yet clear which setup is preferable, but in any case it must be symmetric. In a two-mirror cavity, a design in the order of magnitude $d = R = 10\ cm$ would be reasonable, which would then allow a beam diameter of 1.0 mm and 1.4 mm at the waist and the mirrors, respectively (confocal case, see [6], p. 19.)

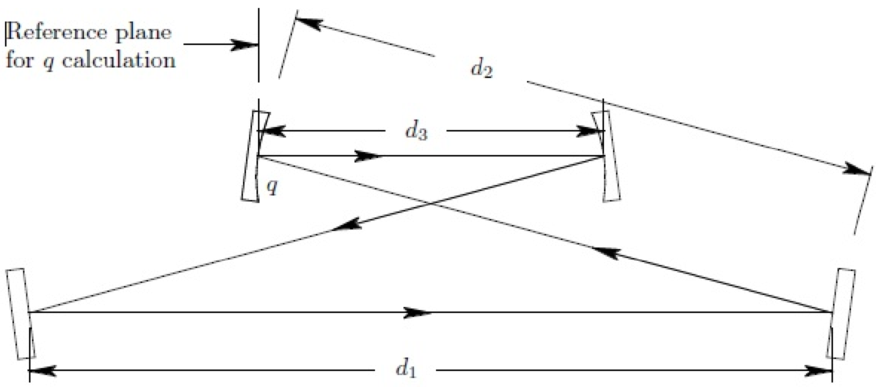

Figure 6: 4-mirror cavity described in [6], p.23.

However, these concepts have to be applied to a four-mirror-cavity, a case which is also treated in [6], p. 23ff. (see fig. 6). The stability condition ([6], eq.(2.44)) to be satisfied is

$$R \leq d_3 \leq \frac{R(d_1 + 2d_2)}{d_1 + 2d_2 - R} \tag{12}$$

Thus, a design with $d_1 = 12.0\ cm$, $d_3 = 8.0 cm$, horizontal distcance $d_h = 2.0\ cm$, resulting in $d_2 = 10.2\ cm$ would meet that criterion, if mirrors with $R = 7.62\ cm$ are used. The effects of astigmatism are not yet considered. A small lateral displacement of the mirrors like 2 cm is desirable (limited by the mirror diameter).

Circular polarization. A standing wave of circular polariation[6] should be realized as follows: Before entering the resonator from both sides, the light passes a filter and is linearly polarized in y-direction. Then, a Faraday rotator should rotate the polarization about $\pm 45°$, depending on an trigger. A subsequent $\frac{\lambda}{4}$-plate transforms the linear polarization in a circular one. Depending on the sign of the angle of the linear polarization, the circular polarization will be left- or righthanded.[7]

Electric measurement. The expected voltage or current should be measurable by commercial high-precision instruments, thus resorting to the Josephson effect with superconducting equipment does not seem necessary at the present stage. However, a lock-in-amplified signal is needed to distinguish the signal from inevitable thermal noise. Furthermore, laser beams always can ionize material of the setup, a source of noise that would easily overlay sublte voltage differences, couldn't the voltage not be reversed by switching the orientation of the circular polarization. Then, a vacuum for the whole setup is likely to be needed and cooling, though not at a superconducting level, should be considered to reduce thermal noise.

Further considerations. Though the discussion is complete so far, for a throrough understanding of the experiment it is useful to link the described setup to known physics. The two metal mirrors in the resonator form nothing else than a capacity. Actually, by dividing the two quantities under consideration above, the current and the increase of the voltage, one obtains $\frac{\epsilon_0 q}{d}$ with the dimensions $\frac{As}{V}$ of a capacity. By generating an electric field between the plates, they will become charged, as it occurs in the case of electric influence. On completely isolated mirrors, the charges on the respective mirror would shift towards the region which is reflecting light. Thus it should be useful in any case to keep the capacity, i.e. the mirror surfaces small. They should be isolated from the rest of the components and linked by a short circuit passing the amplifier.

3.2 Required equipment and open questions

The following equipment will be necessary to carry out the experiment:

- A CW Laser, possibly $Nd - YAK$ at 1064 nm, with power $\approx 4\ W - 8\ W$,
- A beam splitter
- $\frac{\lambda}{4}$- and $\frac{\lambda}{2}$-plates and polarization filter (linear)
- Faraday-rotator capable of switching $\pm 45°$, triggered electrically. Since for terbium gallium garnet (TGG) at 1064 nm, the Verdet constant is $-40\frac{rad}{Tm}$, this would require a magnetic field of $0.65T$ for al length $l = 3.0\ cm$.
- Two identical, plane metal mirrors (silver or gold) with an sufficient resistance to overheating, with reflectance > 0.98.
- Two identical mirrors (0.98) with dielectric coating with an sufficient resistance to overheating, concave (curvature radius $76.2mm$), diameter 1 inch (even smaller), with adjustable position (Piezo-element ?).
- A vacuum chamber of approximate size 20 cm x 20 cm x 50 cm, cooling facilities.
- Voltmeter and/or Amperemeter with pV (pA, better fA) sensitivity, lock-in-amplifier, probably a FEMTO DDPCA-300[8] should be sufficient.

[6] It is interesting that such waves have been realized in the context of the Kapitza-Dirac effect, see [7, 8].

[7] The Faraday rotator will add another $\pm 45°$ turn to the polarization of the reflected light, which is then polarized at right angles to the incoming beam and may be easily filtered out to prevent damage from the beam source.

[8] www.femto.de

Sequence of setup steps.

- Optical 4-cavity, adjustment of optical components, enhancement and finesse test.
- Electrical contacts, current mesurement
- (Optional) Vacuum chamber
- (optional) Cooling

Unresolved questions.

- It is not yet clear how the stability region of the four-mirror-cavity with the proposed curvature radii of the mirrors is affected by astigmatism (see section 3.1).
- It has to be checked whether lenses are necessary for focussing the beam (mode matching, [6], chap. 2.9).
- It has to be checked whether whether the initial beam splitting and polarization causes significant losses of the Laser power (4 W).
- The order of magnitude of thermal effects on the voltage measurement is not well known.
- It has to be checked if the Faraday rotator can produce a 45° twist in the limited space between the mirrors.
- It has to be decided whether the voltage or current measurement is advantageous.
- Which elements have to be placed in the vacuum chamber hast still to be decided.
- It is not clear yet if cooling is needed.

Acknowledgement. The project has been discussed time ago with several people. The author thanks Andreas Maier and Nathaniel Kajumba from LMU Munich for their open-minded interest. Very valuable advice regarding the setup cam from Simon Holzberger and Henning Carstens from LMU's cavity group. Helpful discussions with Karl Fabian and Hannes Hoff are acknowledged.

4 Appendix

Mathematica code:

```
(*from http://en.wikipedia.org/wiki/Rotation_formalisms_in _three _dimensions *)
um = IdentityMatrix[3];(* preliminaries*)
xRot[ph_] := {{1, 0, 0}, {0, Cos[ph], Sin[ph]}, {0, -Sin[ph],  Cos[ph]}};
yRot[ph_] := {{Cos[ph], 0, -Sin[ph]}, {0, 1, 0}, {Sin[ph], 0, Cos[ph]}};
zRot[ph_] := {{Cos[ph], Sin[ph], 0}, {-Sin[ph], Cos[ph], 0}, {0, 0,  1}};
xAlg[p_] := {{0, 0, 0}, {0, 0, p}, {0, -p,
   0}}; (*Lie-Algebra der x-Drehung usw*)
yAlg[p_] := {{0, 0, -p}, {0, 0, 0}, {p, 0, 0}};
zAlg[p_] := {{0, p, 0}, {-p, 0, 0}, {0, 0, 0}};
totangle[m_] := ArcCos[(Tr[m] - 1)/2];
rotVec[m_] := {m[[3, 2]] - m[[2, 3]], m[[1, 3]] - m[[3, 1]],
   m[[2, 1]] - m[[1, 2]]}/(2 Sin[totangle[m]]);
length[vec_] := Sqrt[Apply[Plus, vec^2]];

NonlinEfeld[ampx_, ampy_, ampz_, phiy_, phiz_, nn_, dauer_] :=
```

```
Block[{}, Clear[dt]; verlauf = {};
 veloc = (xAlg[ampx Sin[ 2 Pi t/nn]] +
      ampy yAlg[Sin[ 2 Pi t/nn + phiy]] +
      ampz zAlg[Sin[ 2 Pi t/nn + phiz]])// N ;
  infinis =
   MatrixExp[veloc dt]//N//Simplify;
(*Function of finite rotations in the lie group, generated for an arbitrarly small time step dt*)
  dt = 1/nn; drehs = Table[infinis, {t, 1, dauer nn}] // N // Re;(*
  discretization of the Lie group valued function*)
  res = IdentityMatrix[3];
  For[i = 1, i <= Length[drehs], i++, res = drehs[[i]].res;
   AppendTo[verlauf, res]]; aa = res; Print[rotVec[aa]];
  Print[ "total angle: ", totangle[aa]];
  aa // MatrixForm](* performing the integration*)
```

The results can now be visualized by

```
NonlinEfeld[0.1, 0.1, 0, Pi/2 , 0, 1000, 1];
verl = Map[totangle, verlauf];
ListLinePlot[verl]
```

References

[1] R.P. Feynman, Leighton R.B., and M. Sands. *The Feynman Lectures in Physics*, volume II. California Institute of Technology, 2nd edition, 1963.

[2] L. D. Landau and E. M. Lifshitz. *Theoretical Physics - Classical Field Theory*, volume II. Nauka, Moscow, 1972.

[3] E. Whittaker. *A History of the Theories of Aether and Electricity.* Dover, 1951.

[4] C. F. Frank. On the equations of motion of crystal dislocations. *Proceedings of the Physical Society London*, A 62:131–134, 1949.

[5] http://en.wikipedia.org/wiki/rotation_formalisms_in_three_dimensions.

[6] W. Nagourney. *Quantum Electronics for Atomic Physics.* Oxford University Press, 2010.

[7] P. H. Bucksbaum, D. W. Schumacher, and M. Bashkansky. High-intensity Kapitza-Dirac effect. *Physical Review Letters*, 61:1182–1185, September 1988.

[8] D. L. Freimund, K. Aflatooni, and H. Batelaan. Observation of the Kapitza-Dirac effect. *Nature*, 413:142–143, September 2001.