Galileo’s Solution for the ‘Path of Quickest Descent’

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ABSTRACT

Path of the quickest descent of a material particle from a point A to another point B at a lower lever, in a constant gravitational field, is a famous problem in mathematics. It was solved by Galileo around 1638. Galileo’s solution was that the quickest path is the arc of a circle with B as the lower end of the vertical diameter of the circle in the vertical plane. It is important to note that Galileo does not use the summation of time intervals of travel along the successive chords that connect A and B. He compares the times of travel between two paths from A to B. One path consists of the direct shortest path – the chord connecting A and B. The second path consists of two chords AC and CB. Galileo proves that the two chord path ACB is quicker than the single chord path. Then he compares the two chord path with the three chord path, AC, CD, DB and proves the three chord path ACDB is quicker than the two chord path ACB. He extends this procedure indefinitely to more and more chords and proves that the arc of the circle is the quickest path of travel from A to B.

Later, John Bernoulli solves this problem and poses it as a challenge to peers to solve it. Many well known mathematicians that include Bernoulli’s elder brother Jacob Bernoulli, Newton, Pascal among others solved it. John Bernoulli’s solution was based on Fermat’s least time principle. To account for the path followed by a ray of light between two points, Fermat enunciated the principle of least time. According to this principle light takes the path of minimum time in going from the initial to the final point involving reflections or of refractions on its way. Bernoulli argued that if light follows Fermat’s principle in economizing the time of travel between two points, why not a material particle also follow that principle, so that nature economizes on the number of principles required to govern various processes? Arguing thus, he employed Fermat’s least time principle and arrived at a different solution from that of Galileo’s solution. This solution became very famous and gave rise to many other mathematical developments. In contrast to Galileo’s circular path, Bernoulli’s solution was a ‘brachistochrone’. We discussed Bernoulli’s solution in an article in this journal earlier.

Bernoulli’s solution involves the summation of time intervals of travel along the successive chords of the brachistochrone. Galileo did not add the time intervals because time intervals along paths of different accelerations are not additive – they are additive if and only if they are along path of the same value of acceleration. Students must get a good grasp of this idea in order to appreciate Galileo’s solution. As Erlichson says, this study provides some very interesting information on Galileo’s geometrical methods.

KEY WORDS: Path of quickest descent, Galileo, Brachistochrone

INTRODUCTION

Given two points not lying on the same vertical line what is the path followed by a point mass moving in a constant gravitational field, from the upper to the lower point in the least time? This simple question led to most profound mathematics. The problem came to be known as ‘the brachistochrone’ problem. It was described as the ‘Helen of geometers’ for the charm it held and the mathematical strategies it led to. Two notable solutions to the problem, which are not equivalent, are available. One solution is due to Galileo[1]. This solution gives the circular arc connecting the two points in a vertical plane, the lower point being the lowest point on the circle, to be the quickest path. The other solution is due to Johann Bernoulli. This solution gives brachistochrone as the path of quickest descent.

Much earlier to the famous solution arrived at by John Bernoulli to the problem of the path of quickest descent; Galileo addressed it and arrived at a different solution. Posterity declared Galileo's solution to be erroneous and Bernoulli's solution to be correct.

In an earlier communication[2] we discussed Bernoulli's solution, popularly known as the '*brachistochrone'*. Bernoulli's method was based on Fermat's principle of least time, according to which, light takes the path of least time to travel from one point, say A, to another point, say B. This principle is applied to the path of reflection, when A and B are present in one and the same medium. It is also applied to the path of refraction when A and B are present in two different media. The principle applies even when A and B are separated by a series of media with continuously varying refractive index. In this case light reaches B from A after continuous refractions on the way from point to point. In each and every one of these cases, light takes the least time to travel from A to B according to Fermat.

Fermat's least time principle is said to be a consequence of Snell's laws. According to Fermat's principle, the time of travel, 't' from A to B is given by the sum of times of travel, 't_i' in each medium separating the points A and B and has the least value compared to other paths connecting the two paths.

$$t = \sum t_i \tag{1}$$

The distance travelled S, is the sum of lengths (distances), s_i of the path traversed in different media between A to B.

$$S = \sum S_i \tag{2}$$

The speed of travel, v_i in each medium is different,

$$v_i \neq v_j \tag{3}$$

In the case of reflection, we have

$$v_i = v_j = constant \tag{4}$$

Bernoulli reasoned that if light follows Fermat's principle for its quickest path of travel, a material particle could also follow the same principle for its path of quickest descent. So he fashioned his solution for the path of quickest descent of a material particle to be along a series of planes inclined at different angles to the vertical (the direction of the free fall), on similar lines to the path followed by light in travelling through media with different refractive index values. The planes of different inclinations offer different values of constant accelerations, g_i, along them. As the number of planes increases and tends to infinity, the acceleration continuously increases along the path of descent of the particle. Since Bernoulli's solution is based on Fermat's principle, it involves the summation of times of travel along each of the planes. The sum of times is a minimum for the brachistochrone path.

Galileo's solution, and this is very important to note, does not depend upon the concept of the summation of times of travel along a series of planes of different inclination θ_i to the vertical to obtain the time of travel; no addition of times of travel along planes of different accelerations is involved here. Galileo compares the times along two different paths from a point D to a point C. One path consists of two conjugate chords DB, BC from the point D on the circle to the lowest point C of the circle. The second path consists of a single chord DC from D to C. Galileo shows the two chord path is quicker than the single chord path. He extends the result to the case of a large number of conjugate chords which in the limit tends to the arc of the circle, giving the quickest path of descent. An important message of Galileo's method is that the times of travel with different values of acceleration are not additive quantities. Times of travel are additive if and only if the acceleration is constant all along the path. Galileo's solution is not properly understood. Even as recently as late 1990s, Galileo's method was unfairly criticized and a challenge was thrown[3] to prove a result, supposedly assumed but not proven by Galileo, in his proof.

We discuss in this article the details of Galileo's solution for the path of quickest descent. The material in this article is drawn almost entirely from reference 1.

GALILEO'S SOLUTION[1]

PROPOSITION XXXVI OF 3RD DAY

Galileo's solution is described[1] in Proposition XXXVI of 3rd day of Galileo's "Two New Sciences". He uses three lemmas (which he proves earlier) for the solution of this problem. The proof is based on the geometric construction shown in Figs. 1a, 1b. This is same as figure 102 in reference 1. A particle starts from rest at a point D on the arc DBC, not exceeding a quarter of a circle, and reaches the lowest point 'C' on the circle in the vertical plane.

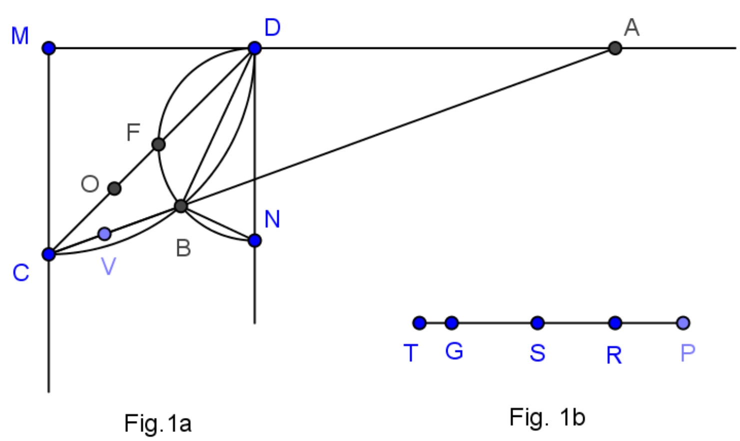

Fig.1a Galileo's geometric construction to obtain the the path of quickest descent of a particle moving in a vertical plane from point D to C along chords of a circle. Fig. 1b. Time scale corresponding to the different distances of descent.

Galileo proves that the time of travel along the broken path of two chords DB and CB (B is a point on the arc between D and C) is less than the time along the single chord DC.

PROOF OF THE PROPOSITIN

Through D, draw the horizontal MDA cutting CB extended, at A. Draw DN and MC perpendicular to MD and BN perpendicular to BD. Draw the circum circle DFBN of the right triangle DBN cutting Dc at F.

Choose point ‘O’ on DC such that DO is the mean proportional between CD and DF. Similarly, choose point ‘V’ on AC such that AV is the mean proportional between CA and AB. The lengths of line segments in Fig. 1a represent distances travelled.

We draw in Fig. 1b, a line on which times of travel are represented. Let the length PS (Fig. 1b) represent the time of descent along the whole distance DC or along BC, both of which require the same time. Choose a point R on SP such that

$$DC : DO \; = time\, PS : \; time\, PR \tag{5}$$

Then PR will represent the time in which a body, starting from rest at D will travel the distance DF, while RS will represent the time in which the remaining distance FC will be travelled.

Since PS is also the time of descent, from rest at B along BC, and if we choose ‘T’ such that,

$$BC : CD = PS : PT \tag{6}$$

then PT will measure the time of descent, from A to C.

We now proceed to show that the time of travel along the broken line of two chords DB and BC is shorter than time along the chord DC. (To prove this Galileo makes use of three lemmas). The proof continues after lemma – 1.

LEMMA – 1

In this lemma Galileo shows FB (see Fig. 2) is the mean proportional of the segments DB, BE of the secant BED (or chord BDE). This figure is same as the figure 99 of Galileo[1].

Let CD be drawn perpendicular to the vertical diameter AB of a circle with B as the lower end. Let CD cut the circle at F. From B draw the line BED at random; draw the line FB. Then FB is the mean

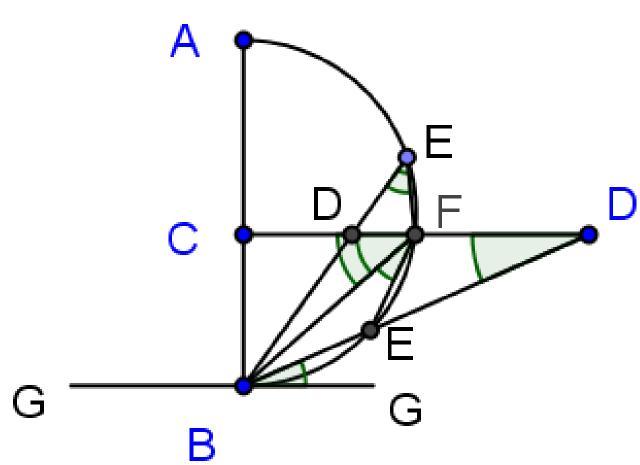

Fig. 2. Lemma - 1

proportional between DB and BE. Join EF. Through B draw tangent BG parallel to CD

$$D\hat{B}G = F\hat{D}B \quad and \quad G\hat{B}E = E\hat{F}B \ (angles\ subtended\ by\ arc\ BE) \tag{7}$$

Triangles FDB and FEB are similar.

$$\therefore \quad BD : BF = BF : BE \tag{8}$$

Therefore BF is the mean proportional between DB and BE

(End of Lemma -1).

From this lemma we find that CD is the mean proportional between AC and CB in Fig 1.

Proof of proposition continues.

We choose point G on line PS such that,

$$CA : AV = PT : PG \tag{9}$$

Then PG will represent the time of descent from A to B, while GT will be the residual time of descent along BC following descent from A to B.

But, since the diameter DN of the circle DFN is a vertical line, the chords DF and DB will be travelled in equal times (see appendix). Hence, if we prove that a body will travel BC, after descent along DB in a shorter time than it will FC after descent along DF then we would have proved the proposition.

But a body descending from D along DB, will traverse BC in the same time as if it had come from A along AB, since the speed acquired is the same in descending along DB as along AB when it reaches B. Therefore, it remains to be shown that descent along BC after AB is quicker than along FC after DF.

We have already shown that GT represents the time along BC after AB as also that RS is the time along FC after DF. Accordingly, it must be shown that RS is greater than GT.

$$Since \quad SP : PR = CD : DO \tag{10}$$

It follows that,

$$RS : SP = OC : CD \tag{11}$$

We also have,

$$SP : PT = DC : CA \tag{12}$$

$$and \ \because \quad TP : PG = CA : AV \tag{13}$$

$$\therefore \quad PT : TG = AC : CV \tag{14}$$

$$and \quad RS : GT = OC : CV \tag{15}$$

Therefore, to show that RS > GT it is enough to show that OC > CV.

To show that OC > CV and time RS > time GT, we have to use Lemmas 2 and 3 given below. Therefore, the continuation of this proof follows at the end of lemmas 2 and 3.

LEMMA – 2

Figure 3 below is same as the Fig. 100 of Galileo[1].

Let AC be a line which is longer than DF and let the ratio of AB to BC be greater than that of DE to EF. Then, the lemma - 2 says, AB is greater than DE.

If AB bears to BC a ratio greater than that of DE to EF, then DE will bear to some length shorter than EF, the same ratio which AB bears to BC. Let us call this length EG; then since,

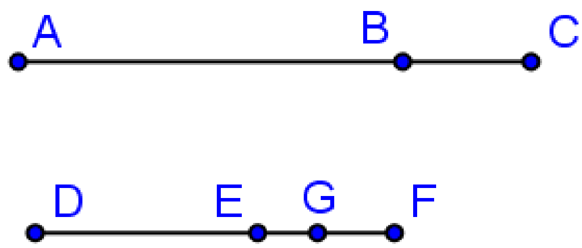

Fig. 3. Lemma - 2

$$AB : BC = DE : EG \tag{16}$$

It follows that
$$CA : AB = GD : DE \tag{17}$$

But, since CA is greater than GD, it follows that BA is greater than ED (end of Lemma – 2).

LEMMA – 3

Let ACIB be the quadrant of a circle (Fig. 4a); from B draw BE parallel to AC. Join CB. About any point in the line BE describe a circle BOES, touching AB at B and intersecting the circumference of the quadrant at I and CB at O. Join the points C and B; draw the line CI, prolonging it to S. Then lemma – 3 says, CI is always less than CO.

Draw the line AI touching the circle BOE. Then, if the line DI be drawn, it will be equal to DB; but, since DB touches the quadrant, DI will also be tangent to it and will be perpendicular to AI. Thus AI touches the circle BOE at I. Since angle AIC is greater than the angle ABC, subtending as it does a larger arc, it follows that the angle SIN is also greater than the angle ABC, Therefore, the arc IES is greater than the arc BO, and the line CS, being nearer the center, is longer than CB. (the bigger the chord, the nearer it is to the center. Or, chord of a bigger arc lies closer to the center than the chord of a smaller arc.) Consequently, CO is greater than CI, since SC : CB = OC : CI. This result is all the more striking if, as in Fig. 4b, the arc BIC is less than a quadrant.

This lemma shows that CF is greater than CB in Fig. 1.

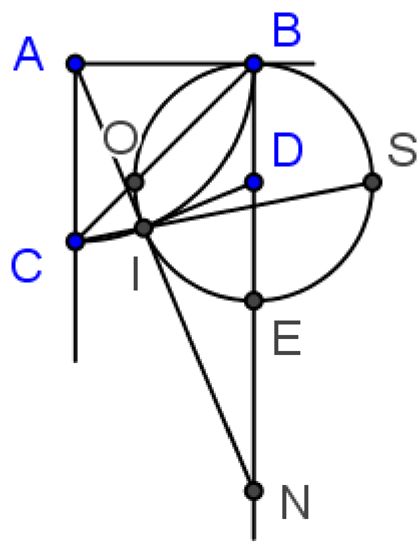

Fig. 4a. Lemma - 3

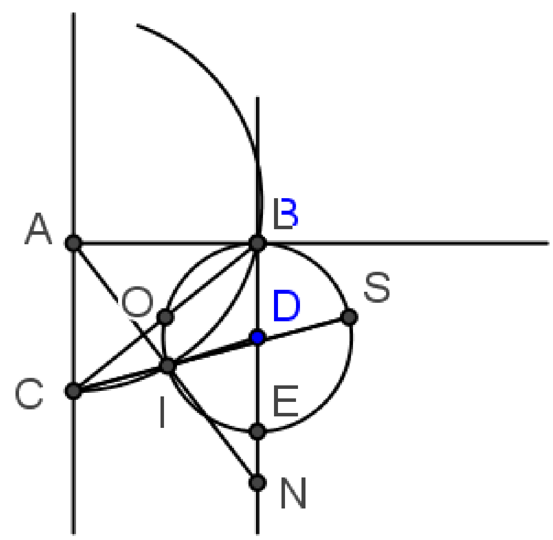

Fig. 4b. Lemma - 3

Proof of proposition continued

Now, we proceed to show that CO is greater than CV (Fig. 1).

Since CF is greater than CB and FD smaller than BA, it follows from lemma -2 that

$$CD:\ DF > CA : AB \tag{18}$$

$$But, \qquad CD:\ DF = CO : OF \tag{19}$$

$$and \qquad CD:\ DO = DO : DF \tag{20}$$

$$CA : AB = CV : VB \tag{21}$$

$$CO : OF > CV : VB \tag{22}$$

According to lemma – 2, we see that,

$$if \quad CO : OF > CV : VB, \quad then\ CO > CV \tag{23}$$

Besides this, it is also clear that, the time of descent along DC : the time of descent along DBC = length DOC : Sum of lengths Do and CV.

SCHOLIUM

Now Galileo proceeds to show in a 'Scholium' (a side note) that the swiftest descent is along the circular

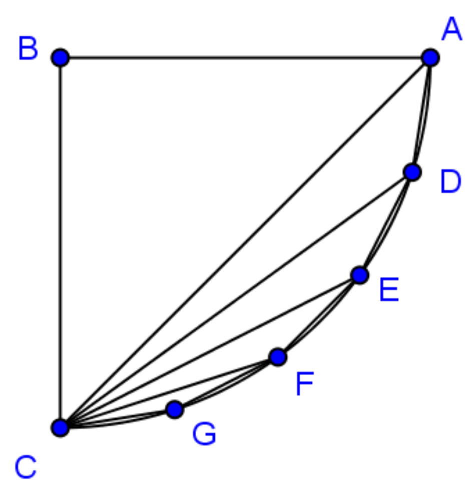

Fig. 5 Extension of the result of single chord versus the two chord path to the many chord path and ulimately to the quickest path of the circular arc.

path and not along the shortest (the straight line joining the initial and lowest points) path.

Let BAEC (Fig. 5) be a quadrant having the side BC vertical. Divide arc AC into any number of equal parts AD, DE, EF, FG, GC, and from C draw straight lines to points A,D,E, F, G and draw straight lines AD, DE, EF, FG, GC.

As shown above, the time of descent along the path ADC is quicker than along AC alone or along DC from rest at D. But a body starting from rest at A will traverse ADC more quickly than the path DC. If it travels along AD starting from rest at A, it will traverse the path DEC in a shorter time than DC alone. Therefore, the time of descent along the three chords path ADEC will take less time than along the two chords path ADC. Similarly, following the descent along ADE the time of descent along EFC is less than that needed for EC alone. Thus, descent is quicker along the four chords path ADEFC than along the three chords path ADEC. Finally, a body after descent along ADEF, will traverse the two chords path FGC more quickly than FC alone. Therefore, along the five chords path ADEFGC, descent will be quicker along the four chords path ADEFC. Consequently, the nearer the inscribed polygon approaches a circle, the shorter is the time required for descent from A to C. This proves the proposition.

What has been proven for the quadrant holds true also for smaller arcs, the reasoning is the same.

ERLICHSON'S CHALLENGE

Relatively recently, Erlichson posed a challenge[3, 4], to prove the so called, 'unproven assumption' in Galileo's statement (translated as), "Yet it seems true that from rest at A, descent is finished more quickly through the two DE-EC than through CD only," preferably by methods available to Galileo.

His challenge was a result of his erroneous assumptions. 1. The premise that the time of descent is the sum of times of travel along the successive chords of the quadrant, and 2. The time of travel along a chord with a finite initial speed v_0 ($\neq 0$) is a function of the angle, θ_i, that the chord makes with the horizontal, in contrast to its independence from θ when the initial speed v_0 is zero. We find the basis of both arguments invalid.

Since Galileo's method does not involve addition of time intervals to get the total time of travel, the first criticism is invalid.

Since the equation used by Erlichson for the time of travel along a chord that makes an angel θ with the horizontal with an initial speed v_0 ($\neq 0$),

$$t_{DI} = \frac{L\ sin\theta}{\left(\frac{v_I + v_0}{2}\right)} \tag{24}$$

is not valid for $\theta = 0$, the second criticism also is invalid; θ has no role to play in the time of travel along any chord including the diameter of the circle (see appendix).

APPENDIX

An important result that is used in the proof of proposition XXXVI is the result of Theorem VI, Proposition VI. It states: "*If from the highest or lowest point in a vertical cycle there be drawn any inclined planes meeting the circumference the times of descent along these chords are each equal to the other.*"

We adapted the following from Mach[4].

We imagine gutters radiating in a vertical plane from a common point A at different degrees of inclination to the horizontal as shown in Fig. A1.We let go simultaneously from point A point masses along the planes. The masses will form a circle at any later instant. After the lapse of a longer time they will be found on a circle of larger radius passing through A. The radii increase proportionately to the squares of time.

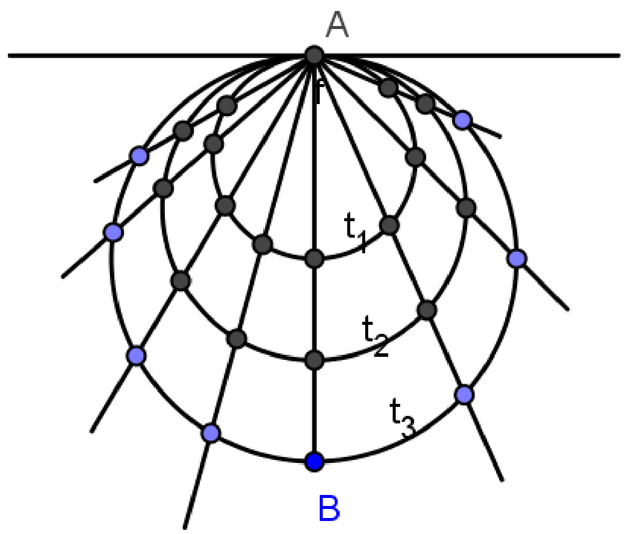

Fig. A1 Diagram showing that masses starting from rest from the upper extrimity of a circle in the vertical plane trvel any and every chord of a given circle in the same interval of time.

From this proposition Galileo deduces a few corollaries. The results are presented in more detail by Mach.

1. The accelerations along the height and the length of an inclined plane are in the inverse ratio of

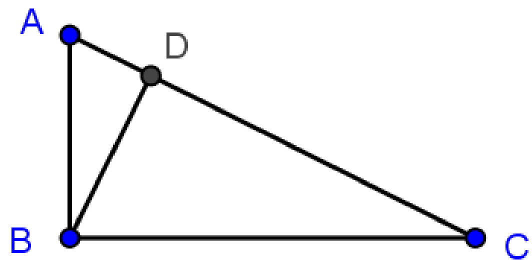

Fig. A2. Diagram showing the distances travelled along the height and length of an inclined plane in equal intervals of time.

height and length. If we cause one body to fall along the length AC of an inclined plane and simultaneously another to fall freely along the height AB, and ask what the distances are that are traversed by the two in equal intervals of time, the solution of the problem will be readily found by simply letting fall from B a perpendicular BD (see Fig. A1) on the length AC.

The part AD thus cut off will be the distance travelled by the body on the inclined plane while the second body is freely falling along the height of the plane.

If we describe a circle in a vertical plane on AB as diameter (Fig. A2), the circle will pass through D, because D is a right angle.

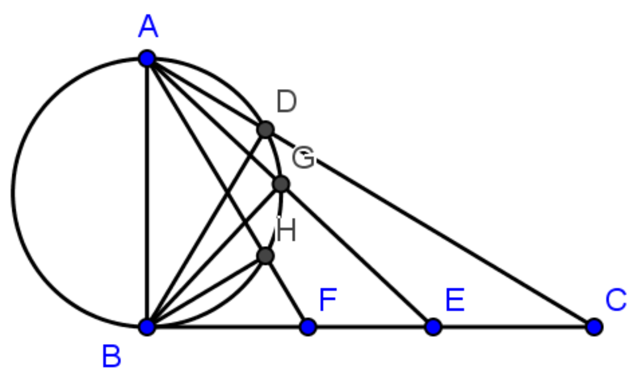

Fig. A3. Diagram showing equality of travel tmes along chords drawn from either extrimity of the diameter of a circle in the vertical plane.

2. Let a number of inclined planes AE, AF, of any degree of inclination, passing through A, and that in every case the chords AG, AH drawn in this circle from the upper extremity of the diameter will be traversed in the same time by a falling body as the vertical diameter itself. Since it is only the length and inclination that are essential here, we may also draw the chords from the lower extremity of the diameter, and say generally: The vertical diameter of a circle is described by a falling body in the same time that any chord through either extremity is so described.

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