Some exercises

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Abstract

The interested reader is invited to solve these exercises.

Exercise 1. With $a,\ b,\ c\ \in\ \ \mathbb{N}$, where $\mathbb{N}$ means $\{1,\ 2,\ 3, \ldots\}$, you conjecture that there are infinitely many solutions to the Diophantine equation $a^3 + b^3 + c^3 = 3abc$. Write a clojure code to get some (feel for) solutions to the equation . Then, persuade yourself to try to prove your conjecture.

Cf. [1].

Exercise 2. Verify the following identity [2].

$$(a^2 + b^2 + c^2 + d^2)(A^2 + B^2 + C^2 + D^2)$$
$$= (aA - bB - cC - dD)^2 + (aB + bA + cD - dC)^2$$
$$+(aC - bD + cA + dB)^2 + (aD + bC - cB + dA)^2.$$

Hint. Use some 'quaternionic tricks' .

Exercise 3. Find all the roots of the equation $16x^5 - 20x^3 + 5x - 1 = 0$. Next, explain why they contain $\sin(\frac{\pi}{10})$.

Hint. First, factor the left-hand side of this equation by $x - 1$.

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Exercise 4. For $a^2 + b^2 = c^2$, $c \neq 0$, show that $(x,\, y) = (\frac{a}{c},\, \frac{ab}{c^2})$ satisfies the equation $x^4 - x^2 + y^2 = 0$ (Gerono) .

Exercise 5. Let the special unitary group SU(2) be represented by $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$, $\alpha,\ \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1$. Next, replace all the entries of this matrix by 2×2 real matrices . Show that such replacement yields an element of special orthogonal group SO(4) .

Hint. Use the following substitutions [3]:

$$\alpha = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \qquad \beta = \begin{pmatrix} C & -D \\ D & C \end{pmatrix}, \qquad A,\, B,\, C,\, D \ \in \ \mathbb{R} \ .$$

Exercise 6. Let A be a 2×2 complex matrix satisfying the following conditions:

1. $\mathrm{tr}\ (A) = 0$;
2. $A^{\dagger} = -A$;
3. $AA^{\dagger} = I_2$.

Explain why A is related to S^2 .

Exercise 7. Suppose the following real matrix

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ -b & c & 0 & -a \\ -c & -b & a & 0 \end{pmatrix}$$

is an element of SO(4) . Explain how it is related to pure unit quaternions by computing $\det\ (A)$.

Exercise 8. Discover for yourself a formula of the solutions to the cubic equation containing $\cosh(x)$ and $\sinh(x)$. Then, evaluate $\sin\frac{\pi}{36}$ to three decimal places using the formula you found, which is henceforth referred to as *formula*, and your favorite software.

Cf. [4].

Exercise 9. Evaluate $\cos\frac{\pi}{36}$ to four decimal places using *formula* and your favorite software.

Exercise 10. Likewise, compute all the roots of the equation $x^3 - x^2 - x - 1 = 0$ [5] to nine decimal places using *formula* and your favorite software.

Exercise 11. Discover for yourself a formula of the solutions to the quartic equation , which is henceforth referred to as **formula**. Then, compute all the roots of the equation $x^4 - x^3 - x^2 - x - 1 = 0$ to ten decimal places using **formula** and your favorite software.

Exercise 12. Show that a 2×2 skew-Hermitian matrix can be represented by the linear-combination of either

$$X^+ = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}, \qquad Y^+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Z^+ = \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix}$$

or

$$X^- = \begin{pmatrix} 0 & -\imath \\ -\imath & 0 \end{pmatrix}, \quad Y^- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z^- = \begin{pmatrix} -\imath & 0 \\ 0 & \imath \end{pmatrix}.$$

N.B. X^+, Y^+, Z^+ correspond to $\imath\ \sigma_i$, whereas X^-, Y^-, Z^- correspond to $\sigma_i\ /\ \imath$, where $i = 1, 2, 3$, respectively.

References

[1] Guy, R. K., "Unsolved problems in number theory. 3rd ed., " Springer-Verlag New York Inc. 2010 **D12**.

[2] Stewart, I. and Tall, D., "Algebraic number theory and Fermat's Last Theorem. 4th ed., " CRC Press 2016 p144.

[3] Dodson, C. T. J. and Parker, P. E.,"A user's guide to algebraic topology," Kluwer Academic Publishers 1997 p250.

[4] Gowers, T., "How to discover for yourself the solution of the cubic" .

[5] Dunlap, R. A., "The golden ratio and Fibonacci numbers, " World Scientific 1997 p61.