Newton's Second Law is Valid in Relativity for Proper Time

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Abstract

In Newtonian single-particle dynamics, time is invariant under inertial transformations, and the particle's speed has no upper bound. In special relativity, it is the particle's proper time, rather than an arbitrary observer's time, which is inertial-transformation invariant, and it is the particle's proper-time speed which has no upper bound. Thus it is reasonable to surmise that the proper-time version of Newton's Second Law is implicit in special-relativistic single-particle dynamics. In fact, gamma times the usual special-relativistic force on a particle equals its rest mass times its proper-time acceleration, and gamma of course goes to unity in the nonrelativistic limit. Furthermore, we show that the scalar potential, the four-vector (electromagnetic) potential and all analogous such tensor potentials, as well as the metric (gravitational) potential, each produces a proper force on the particle equal to its rest mass times its proper-time acceleration. It is to be noted that the Lorentz-covariant four-vector completion of proper-time acceleration, as well as of proper force, has only three components which are mutually independent.

Observed versus proper-length constant velocity of a special relativistic object

An object's constant velocity can be calculated *by dividing the vector segment of its trajectory which it instantaneously intersects by the time it requires to traverse that segment*, but *special-relativistic observed length contraction of that trajectory segment by the factor* γ^{-1} [1], where,

$$\gamma^{-1} \overset{\text{def}}{=} \left(1 - |\dot{\mathbf{r}}/c|^2\right)^{\frac{1}{2}} \leq 1, \tag{1a}$$

implies that the object's special-relativistic observed constant velocity $\dot{\mathbf{r}}$ *equals* γ^{-1} *times its proper-length constant velocity*, i.e., its *proper-length constant velocity* equals $\gamma\dot{\mathbf{r}}$, *whose magnitude* $|\gamma\dot{\mathbf{r}}| = \left(|\dot{\mathbf{r}}|^2/\left(1 - |\dot{\mathbf{r}}/c|^2\right)\right)^{\frac{1}{2}}$ *is unbounded, notwithstanding that* $|\dot{\mathbf{r}}| < c$. Moreover, $\gamma\dot{\mathbf{r}}$ is equal to $(d\mathbf{r}/d\tau)$, *the object's velocity calculated using its Lorentz-invariant differential proper time* $d\tau$, which of course is,

$$d\tau = \left((dt)^2 - |d\mathbf{r}/c|^2\right)^{\frac{1}{2}}. \tag{1b}$$

That $\gamma\dot{\mathbf{r}}$ is equal to $(d\mathbf{r}/d\tau)$ *is a general fact because*,

$$\begin{gathered}\gamma = \left(1/\left(1 - |\dot{\mathbf{r}}/c|^2\right)\right)^{\frac{1}{2}} = \left((dt)^2/\left((dt)^2 - |d\mathbf{r}/c|^2\right)\right)^{\frac{1}{2}} = \left((dt)^2/(d\tau)^2\right)^{\frac{1}{2}} = (dt/d\tau) \;\Rightarrow \\ \gamma\dot{\mathbf{r}} = (dt/d\tau)\dot{\mathbf{r}} = (dt/d\tau)(d\mathbf{r}/dt) = (d\mathbf{r}/d\tau).\end{gathered} \tag{2}$$

An object's Lorentz-transformation invariant differential proper time $d\tau$ *is somewhat analogous to the Galilean-transformation invariant time of Newtonian physics* and, since an object's proper-time speed $|d\mathbf{r}/d\tau| = |\gamma\dot{\mathbf{r}}|$ is unbounded, *that speed is somewhat analogous to the unbounded speed of Newtonian physics*.

The Lorentz-covariant proper-time extension of Newton's Second Law

The usual presentation of single-particle special-relativistic dynamics is,

$$(d\mathbf{p}/dt) = \mathbf{f}, \tag{3a}$$

where $\mathbf{f}$ is the force and the relativistic single-particle momentum $\mathbf{p}$ is given by,

$$\mathbf{p} = m\gamma\dot{\mathbf{r}}, \tag{3b}$$

where m is the particle's rest mass. From Eq. (2) we see that Eq. (3b) can be rewritten,

$$\mathbf{p} = m(d\mathbf{r}/d\tau), \tag{3c}$$

so Eq. (3a) becomes,

$$m(d(d\mathbf{r}/d\tau)/dt) = \mathbf{f}. \tag{3d}$$

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We now multiply the left side of Eq. (3d) by $(dt/d\tau)$ and its right side by γ, as per Eq. (2), which yields,

$$m(d(d\mathbf{r}/d\tau)/dt)(dt/d\tau) = \gamma\mathbf{f}. \tag{3e}$$

We simplify the left side of Eq. (3e) and denote $\gamma\mathbf{f}$ on its right side *as the proper force* $\mathbf{F}$ to obtain,

$$m\big(d^2\mathbf{r}/d\tau^2\big) = \mathbf{F}, \tag{3f}$$

the relativistic extension of Newton's Second Law via proper time. An *example* of Eq. (3f) is the proper force exerted by an electromagnetic field on a particle of charge e, namely,

$$\mathbf{F} = e\gamma(\mathbf{E} + ((\dot{\mathbf{r}}/c) \times \mathbf{B}))\,. \tag{3g}$$

The *fully Lorentz-covariant four-vector completion of* Eq. (3f) *must of course read*,

$$m\big(d^2x^\mu/d\tau^2\big) = F^\mu, \tag{3h}$$

but the nature of proper time *ensures that only three of the four components of the proper force* F^μ *can be mutually independent*. We begin the demonstration of this fact by using Eq. (2) to show that,

$$(dx^\mu/d\tau)\,(dx_\mu/d\tau) = (\dot{x}^\mu\dot{x}_\mu)(dt/d\tau)^2 = (\dot{x}^\mu\dot{x}_\mu)\gamma^2 = \big(c^2 - |\dot{\mathbf{r}}|^2\big)/\big(1 - |\dot{\mathbf{r}}/c|^2\big) = c^2, \tag{3i}$$

which furthermore implies that,

$$\big(d^2x^\mu/d\tau^2\big)(dx_\mu/d\tau) = \tfrac{1}{2}(d((dx^\mu/d\tau)(dx_\mu/d\tau))/d\tau) = \tfrac{1}{2}\big(d\big(c^2\big)/d\tau\big) = 0. \tag{3j}$$

Eq. (3h) together with Eq. (3j) implies that,

$$F^\mu(dx_\mu/d\tau) = m\big(d^2x^\mu/d\tau^2\big)(dx_\mu/d\tau) = 0. \tag{3k}$$

Therefore only three of the four components of the proper force F^μ can be mutually independent. In greater detail, Eq. (3k) together with Eq. (2) yields that,

$$0 = F^\mu(dx_\mu/d\tau) = (F^\mu\dot{x}_\mu)(dt/d\tau) = \big(F^0c - \mathbf{F}\cdot\dot{\mathbf{r}}\big)/\big(1 - |\dot{\mathbf{r}}/c|^2\big)^{\frac{1}{2}}, \text{ which implies that } F^0 = \mathbf{F}\cdot(\dot{\mathbf{r}}/c). \tag{3l}$$

We thus see that F^0 *vanishes altogether in the nonrelativistic limit* $|\dot{\mathbf{r}}/c| \to 0$, for which it is *also* true that $(dt/d\tau) = \gamma = \big(1 - |\dot{\mathbf{r}}/c|^2\big)^{-\frac{1}{2}} \to 1$, so in the nonrelativistic limit Eq. (3h) *reduces to Newton's* $m\ddot{\mathbf{r}} = \mathbf{f}$.

Eq. (3f) shows that the concept of inertial mass, which is *the same* as rest mass, *is just as relevant to relativistic physics as it is to Newtonian physics*. Indeed, the development of Higgs field physics [2] has elaborated the inertial mass concept. An intriguing *inertia issue* is the *existence* of particles, e.g., *photons*, which have *zero inertial mass* (these are asserted *to not couple at all to the Higgs field*). According to Eq. (3c), a zero-inertial-mass particle which has nonzero momentum $|\mathbf{p}| > 0$ *has infinite proper-time speed* because $|d\mathbf{r}/d\tau| = \lim_{m\to 0}(|\mathbf{p}|/m) = \infty$. We now show that this corresponds *to observed speed* $|\dot{\mathbf{r}}|$ *being* c by *inverting* the Eq. (2) relation of proper-time velocity $(d\mathbf{r}/d\tau)$ to observed velocity $\dot{\mathbf{r}}$, namely,

$$(d\mathbf{r}/d\tau) = \gamma\dot{\mathbf{r}} = \big(1 - |\dot{\mathbf{r}}/c|^2\big)^{-\frac{1}{2}}\,\dot{\mathbf{r}}.$$

The *inverse* of this relation is,

$$\dot{\mathbf{r}} = (d\mathbf{r}/d\tau)\,\big(1 + |(d\mathbf{r}/d\tau)/c|^2\big)^{-\frac{1}{2}}, \tag{3m}$$

which has the asymptotic form,

$$\dot{\mathbf{r}} \sim c\,((d\mathbf{r}/d\tau)/|d\mathbf{r}/d\tau|) \;\text{ as }\; |(d\mathbf{r}/d\tau)/c| \to \infty. \tag{3n}$$

This result shows that zero-inertial-mass particles of nonzero momentum $|\mathbf{p}| > 0$, *which therefore have infinite proper-time speed* $|d\mathbf{r}/d\tau| = \lim_{m\to 0}(|\mathbf{p}|/m) = \infty$, consequently *have observed speed* $|\dot{\mathbf{r}}|$ *equal to* c.

We next work out the proper force F^μ exerted on a mass m relativistic particle by a scalar potential $\phi(x^\alpha)$, a four-vector (electromagnetic) potential $A^\nu(x^\alpha)$ and any analogous tensor potential $\Phi^{\nu_1\cdots\nu_n}(x^\alpha)$, as well as by a metric (gravitational) symmetric second-rank tensor dimensionless potential $g_{\mu\nu}(x^\alpha)$. To obtain the equations of motion for these potentials, we construct their relativistic single-particle Lagrangians.

A rest-frame approach to special-relativistic single-particle Lagrangians

A mass m, $\dot{\mathbf{r}} = \mathbf{0}$ special-relativistic particle has energy mc^2 *plus its* $\dot{\mathbf{r}} = \mathbf{0}$ *"rest" potential energy* V_{rest},

$$H_{\text{rest}} = mc^2 + V_{\text{rest}}, \tag{4a}$$

so since the usual Lagrangian term $\dot{\mathbf{r}} \cdot \mathbf{p}$ *vanishes entirely when* $\dot{\mathbf{r}} = \mathbf{0}$, the special-relativistic particle's "rest" Lagrangian L_{rest} and consequent "rest" action S_{rest} are,

$$L_{\text{rest}} = -H_{\text{rest}} = -\big(mc^2 + V_{\text{rest}}\big) \Rightarrow S_{\text{rest}} = -\textstyle\int \big(mc^2 + V_{\text{rest}}\big) dt. \tag{4b}$$

The special-relativistic extension of S_{rest} *is required to be Lorentz invariant*, and therefore is of the form,

$$S_{\text{inv}} = -\textstyle\int \big(mc^2 + V_{\text{inv}}\big) d\tau = -\int \big(mc^2 + V_{\text{inv}}\big)(d\tau/dt)dt, \tag{4c}$$

where $d\tau$ is the particle's Lorentz-invariant differential proper time, and V_{inv} *is its Lorentz-invariant potential energy, which must reduce to* V_{rest} *in the limit* $\dot{\mathbf{r}} \to \mathbf{0}$. Extending V_{rest} to *the Lorentz-invariant* V_{inv} is dealt with case-by-case. Eq. (4c) implies that the full special-relativistic Lagrangian L_{rel} is given by,

$$L_{\text{rel}} = -\big(mc^2 + V_{\text{inv}}\big)(d\tau/dt). \tag{4d}$$

The proper force exerted by a scalar potential

A relativistic particle of mass m which couples to a scalar potential $\phi(x^\alpha)$ with dimensionless coupling strength k has *both* V^ϕ_{rest} *and* V^ϕ_{inv} *equal to* $(k\phi)$, so from Eqs. (4d) and (2),

$$L_\phi = -\big(mc^2 + k\phi\big)(d\tau/dt) = -\big(mc^2 + k\phi\big)\gamma^{-1} = -\big(mc^2 + k\phi\big)\big(1 - |\dot{\mathbf{r}}/c|^2\big)^{\frac{1}{2}}. \tag{5a}$$

Since the generic equation of motion implied by any single-particle Lagrangian L is,

$$d(\partial L/\partial \dot{x}^i)/dt = (\partial L/\partial x^i), \text{ where } i = 1,2,3, \tag{5b}$$

it is very useful in the case of the Lagrangian L_ϕ of Eq. (5a) to note that,

$$\big(\partial\left(\gamma^{-1}\right)/\partial \dot{x}^i\big) = -c^{-2}\gamma\dot{x}^i = -c^{-2}(dt/d\tau)\dot{x}^i = -c^{-2}(dx^i/d\tau). \tag{5c}$$

From Eqs. (5a)–(5c) we obtain that,

$$d\big(\big(m + \big(k\phi/c^2\big)\big)(dx^i/d\tau)\big)/dt = -k(\partial\phi/\partial x^i)(d\tau/dt). \tag{5d}$$

Upon multiplying both sides of Eq. (5d) by $(dt/d\tau)$ and noting that $x^i = -x_i$, it becomes,

$$d\big(\big(m + \big(k\phi/c^2\big)\big)(dx^i/d\tau)\big)/d\tau = k(\partial\phi/\partial x_i), \tag{5e}$$

whose fully Lorentz-covariant four-vector completion clearly is,

$$d\big(\big(m + \big(k\phi/c^2\big)\big)(dx^\mu/d\tau)\big)/d\tau = k(\partial\phi/\partial x_\mu). \tag{5f}$$

Noting that $(d\phi/d\tau) = (\partial\phi/\partial x_\nu)(dx_\nu/d\tau)$, we carry out the outer $d/d\tau$ differentiation on the left side of Eq. (5f) and then move all terms except $m\big(d^2x^\mu/d\tau^2\big)$ to its right side to obtain,

$$m\big(d^2x^\mu/d\tau^2\big) = k\big[(\partial\phi/\partial x_\mu) - \big(1/c^2\big)\big[(dx^\mu/d\tau)(\partial\phi/\partial x_\nu)(dx_\nu/d\tau) + \phi\big(d^2x^\mu/d\tau^2\big)\big]\big] = F^\mu, \tag{5g}$$

where F^μ *is the proper force* $\phi(x^\alpha)$ *exerts on a mass m particle of coupling strength k.*

By applying the identities given by Eqs. (3i) and (3j), namely that,

$$(dx^\mu/d\tau)(dx_\mu/d\tau) = c^2 \text{ and } \big(d^2x^\mu/d\tau^2\big)(dx_\mu/d\tau) = 0, \tag{5h}$$

we verify that the Eq. (5g) proper force F^μ *satisfies the consistency requirement* $F^\mu(dx_\mu/d\tau) = 0$ *of* Eq. (3k).

We also note that if the scalar potential $\phi(x^\alpha)$ is *constant* in x^α, Eq. (5g) implies that,

$$\big(m + \big(k\phi/c^2\big)\big)\big(d^2x^\mu/d\tau^2\big) = 0, \tag{5i}$$

i.e., the particle's *mass m* is effectively *modified by the addition to it of the constant term* $\big(k\phi/c^2\big)$. The Higgs field is thought of as such a constant scalar potential which is able to give an effective mass to otherwise zero-mass particles if they have nonzero dimensionless coupling strength k with that scalar potential [2].

The proper force exerted by a four-vector (electromagnetic) potential

A particle of mass m and charge e at rest in a four-vector electromagnetic potential $A^\nu(x^\alpha)$ has potential energy $V_{\mathrm{rest}}^{A^\nu(x^\alpha)} = eA^0(x^\alpha)$, whose Lorentz-invariant extension $V_{\mathrm{inv}}^{A^\nu(x^\alpha)}$ is given by,

$$V_{\mathrm{inv}}^{A^\nu(x^\alpha)} = (e/c)(dx_\nu/d\tau)A^\nu(x^\alpha).$$

Thus from Eqs. (4d) and (2),

$$L_{A^\nu} = -\big(mc^2 + (e/c)(dx_\nu/d\tau)A^\nu\big)(d\tau/dt) = -mc^2\gamma^{-1} - (e/c)\dot{x}_\nu A^\nu = -mc^2\gamma^{-1} - eA^0 + (e/c)\dot{\mathbf{r}}\cdot\mathbf{A}. \tag{6a}$$

Applying Eqs. (5b) and (5c) to the Eq. (6a) Lagrangian L_{A^ν} yields,

$$d\big(m(dx^i/d\tau) + (e/c)A^i\big)/dt = -(e/c)\dot{x}_\nu\big(\partial A^\nu/\partial x^i\big). \tag{6b}$$

Multiplying both sides of Eq. (6b) by $(dt/d\tau)$ and noting that $x^i = -x_i$ produces,

$$d\big(m(dx^i/d\tau) + (e/c)A^i\big)/d\tau = (e/c)(dx_\nu/d\tau)(\partial A^\nu/\partial x_i), \tag{6c}$$

which we reexpress as,

$$m\big(d^2x^i/d\tau^2\big) = (e/c)\big[(dx_\nu/d\tau)(\partial A^\nu/\partial x_i) - \big(dA^i/d\tau\big)\big]. \tag{6d}$$

Since $\big(dA^i/d\tau\big) = \big(\partial A^i/\partial x_\nu\big)(dx_\nu/d\tau)$, we can rewrite Eq. (6d) as,

$$m\big(d^2x^i/d\tau^2\big) = (e/c)(dx_\nu/d\tau)\big[(\partial A^\nu/\partial x_i) - \big(\partial A^i/\partial x_\nu\big)\big], \tag{6e}$$

whose fully Lorentz-covariant four-vector completion clearly is,

$$m\big(d^2x^\mu/d\tau^2\big) = (e/c)(dx_\nu/d\tau)[(\partial A^\nu/\partial x_\mu) - (\partial A^\mu/\partial x_\nu)] = F^\mu, \tag{6f}$$

where *F^μ is the proper force $A^\mu(x^\alpha)$ exerts on a mass m particle of charge e*. The proper force F^μ *satisfies the consistency requirement $F^\mu(dx_\mu/d\tau) = 0$ of* Eq. (3k) because $(dx_\nu/d\tau)(dx_\mu/d\tau)$ is *symmetric* under interchange of ν and μ, whereas $[(\partial A^\nu/\partial x_\mu) - (\partial A^\mu/\partial x_\nu)]$ is *antisymmetric* under that interchange. Eq. (6f) also implies Eq. (3g), since for $\mu = i = 1$, 2, or 3,

$$\begin{gathered} F^i = (e/c)(\gamma\dot{x}_\nu)\big[(\partial A^\nu/\partial x_i) - \big(\partial A^i/\partial x_\nu\big)\big] = \\ e\gamma\big[-\big(\partial A^0/\partial x^i\big) - (1/c)\dot{A}^i\big] + (e/c)\gamma\sum_{j=1}^{3}\big(\dot{x}^j\big)\big[\big(\partial A^j/\partial x^i\big) - \big(\partial A^i/\partial x^j\big)\big] = \\ e\gamma\big(-\big(\nabla_{\mathbf{r}}A^0\big) - (1/c)\dot{\mathbf{A}}\big)^i + (e/c)\gamma\left((\nabla_{\mathbf{r}}(\dot{\mathbf{r}}\cdot\mathbf{A})) - ((\dot{\mathbf{r}}\cdot\nabla_{\mathbf{r}})\mathbf{A})\right)^i = \\ e\gamma\left(\mathbf{E} + ((\dot{\mathbf{r}}/c)\times(\nabla_{\mathbf{r}}\times\mathbf{A}))\right)^i = e\gamma\left(\mathbf{E} + ((\dot{\mathbf{r}}/c)\times\mathbf{B})\right)^i. \end{gathered} \tag{6g}$$

The proper force exerted by a broad class of Lorentz-invariant potential energies

For *a tensor potential* $\Phi^{\nu_1\cdots\nu_n}(x^\alpha)$, the Lorentz-invariant potential energy $V_{\mathrm{inv}}^{\Phi^{\nu_1\cdots\nu_n}(x^\alpha)}$ *analogous* to the Lorentz-invariant potential energy $V_{\mathrm{inv}}^{A^\nu(x^\alpha)}$ given *above* Eq. (6a) *for the four-vector potential* $A^\nu(x^\alpha)$ is,

$$V_{\mathrm{inv}}^{\Phi^{\nu_1\cdots\nu_n}(x^\alpha)} = K(dx_{\nu_1}/d\tau)\cdots(dx_{\nu_n}/d\tau)\Phi^{\nu_1\cdots\nu_n}(x^\alpha).$$

This Lorentz-invariant potential energy $V_{\text{inv}}^{\Phi^{\nu_1\cdots\nu_n}(x^\alpha)}$ is one *of a broad class* ("bc") *of Lorentz-invariant potential energies* $V_{\text{inv}}^{\text{bc}}((dx_\nu/d\tau), x^\alpha)$ *which are functions of the particle's proper four-velocity* $(dx_\nu/d\tau)$ *and its space-time location* x^α. The special-relativistic single-particle Lagrangian L_{bc} that corresponds to $V_{\text{inv}}^{\text{bc}}$ is,

$$L_{\text{bc}} = -\big(mc^2 + V_{\text{inv}}^{\text{bc}}((dx_\nu/d\tau), x^\alpha)\big)\,\gamma^{-1}. \tag{7a}$$

Eq. (5c) tells us that,

$$\big(\partial\left(\gamma^{-1}\right)/\partial\dot{x}^i\big) = -c^{-2}(dx^i/d\tau), \tag{7b}$$

and it is similarly the case that,

$$(\partial\gamma/\partial\dot{x}^i) = c^{-2}(dx^i/d\tau)\gamma^2, \tag{7c}$$

from which we obtain,

$$(\partial(dx_\nu/d\tau)/\partial\dot{x}^i) = (\partial(\gamma\dot{x}_\nu)/\partial\dot{x}^i) = \big(c^{-2}(dx^i/d\tau)(dx_\nu/d\tau) - \delta_{\nu i}\big)\,\gamma. \tag{7d}$$

We now use Eqs. (7b) and (7d) together with the Eq. (7a) Lagrangian L_{bc} to calculate $(\partial L_{\text{bc}}/\partial\dot{x}^i)$,

$$\begin{aligned}(\partial L_{\text{bc}}/\partial\dot{x}^i) = \big((dx^i/d\tau)\left(m + c^{-2}V_{\text{inv}}^{\text{bc}}\right) - \big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\nu/d\tau)\big)\left(c^{-2}(dx^i/d\tau)(dx_\nu/d\tau) - \delta_{\nu i}\right)\big) = \\ \big((dx^i/d\tau)\left(m + c^{-2}\left(V_{\text{inv}}^{\text{bc}} - \left(\big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\nu/d\tau)\big)(dx_\nu/d\tau)\right)\right)\right) + \big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_i/d\tau)\big)\big).\end{aligned} \tag{7e}$$

Eq. (7a) also yields,

$$(\partial L_{\text{bc}}/\partial x^i) = -\big(\partial V_{\text{inv}}^{\text{bc}}/\partial x^i\big)\,\gamma^{-1} = \big(\partial V_{\text{inv}}^{\text{bc}}/\partial x_i\big)\,(d\tau/dt), \tag{7f}$$

where we have used $\gamma^{-1} = (d\tau/dt)$ and $x_i = -x^i$.

Upon multiplying the equation of motion $d(\partial L_{\text{bc}}/\partial\dot{x}^i)/dt = (\partial L_{\text{bc}}/\partial x^i)$ through by $(dt/d\tau)$, it becomes,

$$d(\partial L_{\text{bc}}/\partial\dot{x}^i)/d\tau = (\partial L_{\text{bc}}/\partial x^i)(dt/d\tau) = \partial V_{\text{inv}}^{\text{bc}}/\partial x_i, \tag{7g}$$

where the last equality is from Eq. (7f). We next insert the Eq. (7e) $(\partial L_{\text{bc}}/\partial\dot{x}^i)$ into Eq. (7g) to obtain,

$$d\big((dx^i/d\tau)\big(m + c^{-2}\big(V_{\text{inv}}^{\text{bc}} - \big(\big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\nu/d\tau)\big)(dx_\nu/d\tau)\big)\big)\big) + \big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_i/d\tau)\big)\big)/d\tau = \partial V_{\text{inv}}^{\text{bc}}/\partial x_i, \tag{7h}$$

whose Lorentz-covariant four-vector completion is clearly,

$$d\big((dx^\mu/d\tau)\big(m + c^{-2}\big(V_{\text{inv}}^{\text{bc}} - \big(\big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\nu/d\tau)\big)(dx_\nu/d\tau)\big)\big)\big) + \big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\mu/d\tau)\big)\big)/d\tau = \partial V_{\text{inv}}^{\text{bc}}/\partial x_\mu, \tag{7i}$$

from which we obtain the proper force F^μ,

$$\begin{aligned}m(d^2x^\mu/d\tau^2) = F^\mu = \; -c^{-2}(d^2x^\mu/d\tau^2)\big(V_{\text{inv}}^{\text{bc}} - \big(\big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\nu/d\tau)\big)(dx_\nu/d\tau)\big)\big) \\ -c^{-2}(dx^\mu/d\tau)d\big(V_{\text{inv}}^{\text{bc}} - \big(\big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\nu/d\tau)\big)(dx_\nu/d\tau)\big)\big)/d\tau - d\big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\mu/d\tau)\big)/d\tau + \big(\partial V_{\text{inv}}^{\text{bc}}/\partial x_\mu\big).\end{aligned} \tag{7j}$$

We next calculate the value of $F^\mu(dx_\mu/d\tau)$ which is implied by the *second* equality of Eq. (7j), noting from Eqs. (3i) and (3j) that $(dx^\mu/d\tau)(dx_\mu/d\tau) = c^2$ and $(d^2x^\mu/d\tau^2)(dx_\mu/d\tau) = 0$, so,

$$\begin{aligned}F^\mu(dx_\mu/d\tau) = \; -d\big(V_{\text{inv}}^{\text{bc}} - \big(\big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\nu/d\tau)\big)(dx_\nu/d\tau)\big)\big)/d\tau \\ -\big(d\big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\mu/d\tau)\big)/d\tau\big)(dx_\mu/d\tau) + \big(\partial V_{\text{inv}}^{\text{bc}}/\partial x_\mu\big)(dx_\mu/d\tau).\end{aligned} \tag{7k}$$

We expand the *first* of the *three terms* on the right side of Eq. (7k) by calculating *both* that,

$$-d\big(V_{\text{inv}}^{\text{bc}}\big)/d\tau = \; -\big(\partial V_{\text{inv}}^{\text{bc}}/\partial x_\mu\big)(dx_\mu/d\tau) - \big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\nu/d\tau)\big)\big(d^2x_\nu/d\tau^2\big). \tag{7l}$$

and *also* that,

$$\begin{aligned}d\big(\big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\nu/d\tau)\big)(dx_\nu/d\tau)\big)/d\tau = \\ \big(d\big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\mu/d\tau)\big)/d\tau\big)(dx_\mu/d\tau) + \big(\partial V_{\text{inv}}^{\text{bc}}/\partial(dx_\nu/d\tau)\big)\big(d^2x_\nu/d\tau^2\big).\end{aligned} \tag{7m}$$

Adding Eq. (7m) *to* Eq. (7l) yields the following result for *the first term* on the right side of Eq. (7k),

$$-\,d\big(V^{\rm bc}_{\rm inv} - \big(\big(\partial V^{\rm bc}_{\rm inv}/\partial(dx_\nu/d\tau)\big)(dx_\nu/d\tau)\big)\big)/d\tau =$$
$$\big(d\big(\partial V^{\rm bc}_{\rm inv}/\partial(dx_\mu/d\tau)\big)/d\tau\big)(dx_\mu/d\tau) - \big(\partial V^{\rm bc}_{\rm inv}/\partial x_\mu\big)(dx_\mu/d\tau), \tag{7n}$$

which when substituted into the right side of Eq. (7k) yields,

$$F^\mu(dx_\mu/d\tau) = 0, \tag{7o}$$

so *the broad class of proper forces* F^μ *given by* Eq. (7j) *all satisfy the* Eq. (3k) *consistency requirement*.

The proper force exerted by a metric (gravitational) potential

A sufficiently simple special-relativistic dynamical system is coupled to a dimensionless symmetric-tensor metric potential $g_{\mu\nu}(x^\alpha)$ *by substituting* $g_{\mu\nu}(x^\alpha)$ *for occurrences of the Minkowski metric tensor* $\eta_{\mu\nu}$ *in the system's Lorentz-invariant action*. To study a single particle's interaction with $g_{\mu\nu}(x^\alpha)$ *only*, the Lorentz-invariant action in which occurrences of $\eta_{\mu\nu}$ are replaced by $g_{\mu\nu}(x^\alpha)$ *must be that of the free particle, i.e.*,

$$S_{\rm free} = -\int mc^2 d\tau, \tag{8a}$$

whose Lorentz-invariant proper differential time $d\tau$ is given by Eq. (1b),

$$d\tau = \big((dt)^2 - |d\mathbf{r}/c|^2\big)^{\frac{1}{2}} = (dx^\mu dx_\mu)^{\frac{1}{2}}/c = (dx^\mu \eta_{\mu\nu} dx^\nu)^{\frac{1}{2}}/c. \tag{8b}$$

Having expressed $d\tau$ in terms of $\eta_{\mu\nu}$, we replace $\eta_{\mu\nu}$ by $g_{\mu\nu}(x^\alpha)$, which changes the Eq. (8a) free-particle action $S_{\rm free}$ to the following action $S_{g_{\mu\nu}}$ for the interaction of the particle with the metric potential $g_{\mu\nu}(x^\alpha)$,

$$S_{g_{\mu\nu}} = -\int mc(dx^\mu g_{\mu\nu} dx^\nu)^{\frac{1}{2}} = -\int mc(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}} dt,$$

from which the Lagrangian $L_{g_{\mu\nu}}$ for the interaction of the particle with $g_{\mu\nu}(x^\alpha)$ follows,

$$L_{g_{\mu\nu}} = -mc(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}} = -mc\Big(\dot{x}^0 g_{00}\dot{x}^0 + 2\dot{x}^0\textstyle\sum_{j=1}^3 g_{0j}\dot{x}^j + \sum_{j=1}^3\sum_{k=1}^3 \dot{x}^j g_{jk}\dot{x}^k\Big)^{\frac{1}{2}}, \text{ where } \dot{x}^0 = c. \tag{8c}$$

Since $L_{g_{\mu\nu}}$ is given in the observer's time t, presenting its equation of motion in the particle's proper time τ requires $(d\tau/dt)$—we saw above that the particle's coupling to $g_{\mu\nu}(x^\alpha)$ *changed* $d\tau$ *from* $(dx^\mu\eta_{\mu\nu}dx^\nu)^{\frac{1}{2}}/c$ *to* $(dx^\mu g_{\mu\nu}dx^\nu)^{\frac{1}{2}}/c$, which implies that that coupling *correspondingly changed* $(d\tau/dt)$ *from* $(\dot{x}^\mu\eta_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}}/c$ *to*,

$$(d\tau/dt)_{g_{\mu\nu}} \stackrel{\rm def}{=} (\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}}/c. \tag{8d}$$

A crucial physics-related restriction on $g_{\mu\nu}(x^\alpha)$ is that for all x^α, *its four matrix eigenvalues are required have the same signs as the matrix eigenvalues of* $\eta_{\mu\nu}$, *namely* $\{+,-,-,-\}$ [3]. Therefore for all x^α, $g_{\mu\nu}(x^\alpha)$ *has a matrix inverse*, which is conventionally denoted $g^{\lambda\kappa}(x^\alpha)$. Thus, for example,

$$g^{\lambda\kappa}(x^\alpha)g_{\kappa\nu}(x^\alpha) = \delta^\lambda_\nu. \tag{8e}$$

Before we work out the equation of motion implied by the Eq. (8c) Lagrangian $L_{g_{\mu\nu}}$, we note *the generalizations of the proper-velocity and proper-acceleration identities given by* Eqs. (3i) *and* (3j) *that ensue in the presence of a metric potential* $g_{\mu\nu}(x^\alpha)$. The easily-guessed generalization of the Eq. (3i) identity is,

$$(dx^\mu/d\tau)g_{\mu\nu}(dx^\nu/d\tau) = (\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)/((d\tau/dt)_{g_{\mu\nu}})^2 = c^2, \tag{8f}$$

where the last equality follows from Eq. (8d). The Eq. (3j) identity's generalization is then obtained via differentiation with respect to τ of the Eq. (8f) identity,

$$0 = d\left(c^2\right)/d\tau = d((dx^\mu/d\tau)g_{\mu\nu}(dx^\nu/d\tau))/d\tau =$$
$$2\big(d^2x^\mu/d\tau^2\big)g_{\mu\nu}(dx^\nu/d\tau) + (dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau), \tag{8g}$$

which implies the following generalization of the Eq. (3j) identity,

$$\big(d^2x^\lambda/d\tau^2\big)g_{\lambda\gamma}(dx^\gamma/d\tau) = -\tfrac{1}{2}(dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau). \tag{8h}$$

Given this identity, a purported proper force F^λ on a particle of mass m that is claimed to adhere to,

$$m\big(d^2x^\lambda/d\tau^2\big) = F^\lambda, \tag{8i}$$

must be such that it satisfies the consistency requirement,

$$F^\lambda g_{\lambda\gamma}(dx^\gamma/d\tau) = -\tfrac{1}{2}m(dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau). \tag{8j}$$

We now work out the equation of motion implied by the Eq. (8c) Lagrangian $L_{g_{\mu\nu}} = -mc(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}}$,

$$\begin{gathered}(\partial L_{g_{\mu\nu}}/\partial\dot{x}^i) = -\tfrac{1}{2}mc\Big(2g_{i0}\dot{x}^0 + 2\textstyle\sum_{j=1}^3 g_{ij}\dot{x}^j\Big)/(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}} = \\ -mc(g_{i\nu}\dot{x}^\nu)/(c(d\tau/dt)_{g_{\mu\nu}}) = -mg_{i\nu}(dx^\nu/d\tau),\end{gathered} \tag{8k}$$

and,

$$\begin{gathered}(\partial L_{g_{\mu\nu}}/\partial x^i) = -\tfrac{1}{2}mc(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)\dot{x}^\nu)/(\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu)^{\frac{1}{2}} = \\ -\tfrac{1}{2}mc(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)\dot{x}^\nu)/(c(d\tau/dt)_{g_{\mu\nu}}) = -\tfrac{1}{2}m(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)(dx^\nu/d\tau)).\end{gathered} \tag{8l}$$

Inserting the Eq. (8k) and (8l) results into the generic Eq. (5b) equation of motion produces,

$$-m(d(g_{i\nu}(dx^\nu/d\tau))/dt) = -\tfrac{1}{2}m(\dot{x}^\mu(\partial g_{\mu\nu}/\partial x^i)(dx^\nu/d\tau)). \tag{8m}$$

After dividing both sides of Eq. (8m) by $(d\tau/dt)_{g_{\mu\nu}}$, this equation of motion can be reexpressed as,

$$-m(d(g_{i\nu}(dx^\nu/d\tau))/d\tau) = -\tfrac{1}{2}m((dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^i)(dx^\nu/d\tau)). \tag{8n}$$

The four-vector completion of Eq. (8n) clearly is,

$$-m(d(g_{\kappa\nu}(dx^\nu/d\tau))/d\tau) = -\tfrac{1}{2}m((dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\nu/d\tau)). \tag{8o}$$

Carrying out the outer $d/d\tau$ differentiation on the left side of Eq. (8o) yields two terms,

$$-mg_{\kappa\nu}\big(d^2x^\nu/d\tau^2\big) - m((dx^\mu/d\tau)(\partial g_{\kappa\nu}/\partial x^\mu)(dx^\nu/d\tau)) = -\tfrac{1}{2}m((dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\nu/d\tau)). \tag{8p}$$

Because $((dx^\mu/d\tau)(dx^\nu/d\tau))$ is *symmetric* under interchange of μ and ν, Eq. (8p) can be rewritten as,

$$-mg_{\kappa\nu}\big(d^2x^\nu/d\tau^2\big) = \tfrac{1}{2}m((dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau)). \tag{8q}$$

Making use of Eq. (8e), we multiply both sides of Eq. (8q) by $-g^{\lambda\kappa}$ and sum over the index κ to obtain,

$$m\big(d^2x^\lambda/d\tau^2\big) = -\tfrac{1}{2}mg^{\lambda\kappa}(dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau) = F^\lambda, \tag{8r}$$

where F^λ *is the proper force* $g_{\mu\nu}(x^\alpha)$ *exerts on a mass* m *particle*.

To check that F^λ satisfies the consistency requirement of Eq. (8j), we use $g^{\lambda\kappa} = g^{\kappa\lambda}$ to rewrite F^λ as,

$$F^\lambda = -\tfrac{1}{2}m(dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau)g^{\kappa\lambda}, \tag{8s}$$

which, since $g^{\kappa\lambda}g_{\lambda\gamma} = \delta^\kappa_\gamma$, yields that,

$$\begin{gathered}F^\lambda g_{\lambda\gamma}(dx^\gamma/d\tau) = -\tfrac{1}{2}m(dx^\mu/d\tau)[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)](dx^\nu/d\tau)(dx^\kappa/d\tau) = \\ -\tfrac{1}{2}m(dx^\mu/d\tau)(\partial g_{\mu\nu}/\partial x^\kappa)(dx^\kappa/d\tau)(dx^\nu/d\tau),\end{gathered} \tag{8t}$$

as required by Eq. (8j), where the last equality ensues after appropriately renaming contracted indices.

It is to be noted that Eq. (8r) *is conventionally written using the Christoffel symbol* $\Gamma^\lambda_{\mu\nu}$, i.e. [4],

$$\begin{gathered}\big(d^2x^\lambda/d\tau^2\big) + (dx^\mu/d\tau)\Gamma^\lambda_{\mu\nu}(dx^\nu/d\tau) = 0 \;\text{ where,} \\ \Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}g^{\lambda\kappa}[(\partial g_{\kappa\nu}/\partial x^\mu) + (\partial g_{\kappa\mu}/\partial x^\nu) - (\partial g_{\mu\nu}/\partial x^\kappa)].\end{gathered} \tag{8u}$$

References

[1] Length contraction–Wikipedia, https://en.wikipedia.org/wiki/Length_contraction.

[2] Higgs field–Simple Wikipedia, https://simple.wikipedia.org/wiki/Higgs_field.

[3] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972), Section 3.6, pp. 85–86.

[4] S. Weinberg, op. cit., Eq. (3.2.3), p. 71 and Eq. (3.3.7), p. 75.