What is Anomalous about the Anomalous Magnetic Moment?

Jean Louis Van Belle, *Drs*, *MAEc*, *BAEc*, *BPhil*
14 December 2018

Email: jeanlouisvanbelle@outlook.com

Abstract: This paper is a very short didactic exploration of the geometry of the experiments measuring the anomalous magnetic moment. It is argued that there is nothing anomalous about it. The Larmor precession invalidates the usual substitution that is made for the gyromagnetic ratio of the precessional motion. In fact, if the substitution is made, one gets a value of −1/2 instead of zero. We should, therefore, not wonder why the anomalous magnetic moment is not equal to zero, but why it is so *nearly* zero.

Text: What is referred to as the electron's anomalous magnetic moment is actually *not* a magnetic moment. It is just some (real) number: it does not have a physical dimension – as opposed to, say, an actual magnetic moment, which – in the context of quantum mechanics – is measured in terms of the Bohr magneton $\mu_B = q_e\hbar/2m \approx 9.274\times10^{-24}$ *joule per tesla*.[1] To be precise, the electron's anomalous magnetic moment – denoted by a_e – is usually *defined* as the (half-)difference between (1) a supposedly *real* gyromagnetic ratio (g_e) and (2) Dirac's theoretical value for the gyromagnetic ratio of a spin-only electron (g = 2)[2]:

$$a_e = \frac{g_e - g}{2} = \frac{g_e - 2}{2} = \frac{g_e}{2} - 1$$

It is weird to use the g-factor for a spin-only electron, because the electron in the cyclotron (a Penning trap) is actually *not* a spin-only electron: it follows an orbital motion – as we will explain shortly. It is also routinely said (and written) that its *measured* value is 0.00115965218085(76). The 76 (between brackets) is the (un)certainty: it is equal to 0.00000000000076, i.e. 76 *parts per trillion* (ppt) and it is measured as a standard deviation.[3] However, the experiments do *not* directly measure a_e. What is actually being measured in the *Penning traps* that are used in these experiments are two slightly different frequencies – an orbital frequency and a precession frequency, to be precise – and a_e is then *calculated* as the fractional difference between the two:

---

[1] Needless to say, the tesla is the SI unit for the magnitude of a magnetic field. We can also write it as [**B**] = N/(m·A), using the SI unit for current, i.e. the ampere (A). Now, 1 C = 1 A·s and, hence, 1 N/(m·A) = 1 (N/C)/(m/s). Hence, the physical dimension of the magnetic field is the physical dimension of the electric field (N/C) divided by m/s. We like the [**E**] = [**B**]·m/s expression because it reflects the *geometry* of the electric and magnetic field vectors.

[2] See: Physics Today, 1 August 2006, p. 15 (https://physicstoday.scitation.org/doi/10.1063/1.2349714). The article also explains the methodology of the experiment in terms of the frequency measurements, which we explain above.

[3] See: G. Gabrielse, D. Hanneke, T. Kinoshita, M. Nio, and B. Odom, New Determination of the Fine Structure Constant from the Electron g Value and QED, Phys. Rev. Lett. 97, 030802 (2006). More recent theory and experiments may have come up with an even more precise number.

$$a_e = \frac{f_p - f_c}{f_c}$$

Let us go through the motions here – literally. The orbital frequency f_c is the cyclotron frequency: a charged particle in a Penning trap will move in a circular orbit whose frequency depends on the charge, its mass and the strength of the magnetic field *only*. We write:

$$f_c = \frac{1}{2\pi} \cdot \frac{\mathrm{q}}{\mathrm{m}} \cdot \mathrm{B}$$

The subscript *c* stands for *c*yclotron – or *c*ircular, if you want. We should not think of the speed of light here! In fact, the orbital velocity is a (relatively small) fraction of the speed of light and we can, therefore, use non-relativistic formulas. The derivation of the formula is quite straightforward – but we find it useful to recap it. It is based on a simple analysis of the Lorentz force, which is just the magnetic force here: $\mathbf{F} = \mathrm{q}(\boldsymbol{v} \times \mathbf{B})$.[4] Note that the frequency does *not* depend on the velocity or the radius of the circular motion. This is actually the whole idea of the trap: the electron can be inserted into the trap with a precise kinetic energy and will follow a circular trajectory if the frequency of the alternating voltage is kept constant. This is why we italicized *only* when writing that the orbital frequency depends on the charge, the mass and the strength of the magnetic field *only*. So what is the derivation? The Lorentz force is equal to the centripetal force here. We can therefore write:

$$\mathrm{q} \cdot v \cdot \mathrm{B} = \frac{mv^2}{r}$$

The v^2/r factor is the centripetal acceleration. Hence, the F = m·v^2/r does effectively represent Newton's force law. The equation above yields the following formula for v and the v/r ratio:

$$v = \frac{\mathrm{q} \cdot r \cdot \mathrm{B}}{\mathrm{m}} \Rightarrow \frac{v}{r} = \frac{\mathrm{q} \cdot \mathrm{B}}{\mathrm{m}}$$

Now, the cyclotron frequency f_c will respect the following equation:

$$v = \omega \cdot r = 2\pi \cdot f_c \cdot r$$

Re-arranging and substituting v for q·r·b/m yields:

$$f_c = \frac{v}{2\pi \cdot r} = \frac{\mathrm{q} \cdot \mathrm{B}}{2\pi \cdot \mathrm{m}}$$

The associated *current* will be equal to:

$$\mathrm{I} = \mathrm{q} \cdot f = \mathrm{q}\frac{v}{2\pi \cdot r} = \frac{\mathrm{q}^2 \cdot \mathrm{B}}{2\pi \cdot \mathrm{m}}$$

Hence, the magnetic moment is equal to:

$$\mu = \mathrm{I} \cdot \pi \cdot r^2 = \mathrm{q}\frac{v}{2\pi \cdot r} \cdot \pi \cdot r^2 = \frac{\mathrm{q} \cdot v \cdot r}{2}$$

---

[4] Our derivation is based on the following reference: https://www.didaktik.physik.uni-muenchen.de/elektronenbahnen/en/b-feld/anwendung/zyklotron2.php.

The angular momentum – which we will denote by – is equal to[5]:

$$\mathrm{J} = I \cdot \omega = \mathrm{m}r^2 \cdot \frac{v}{r} = \mathrm{m} \cdot r \cdot v$$

Hence, we can write the g-factor as:

$$g_c = \frac{2\mathrm{m}}{\mathrm{q}}\frac{\mu}{\mathrm{J}} = \frac{2\mathrm{m}}{\mathrm{q}} \cdot \frac{\mathrm{q} \cdot v \cdot r}{2\mathrm{m} \cdot r \cdot v} = 1$$

It is what we would expect it to be: it is the gyromagnetic ratio for the *orbital* moment of the electron. It is *one*, not 2. Because g_c is 1, we can write something very obvious:

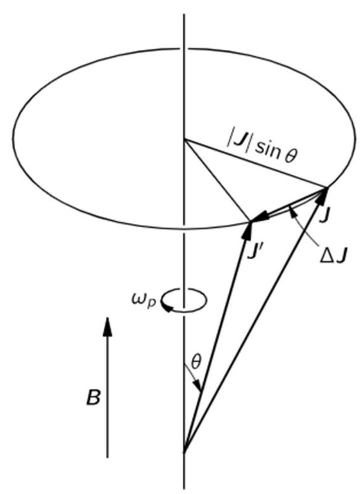
$$f_c = g_c \frac{\mathrm{q} \cdot \mathrm{B}}{2\pi \cdot \mathrm{m}}$$

We should also note another equality here:

$$\frac{2\mathrm{m}}{\mathrm{q}}\frac{\mu}{\mathrm{J}} = 1 \Leftrightarrow \frac{\mu}{\mathrm{J}} = \frac{\mathrm{q}}{2\mathrm{m}}$$

Let us now look at the other frequency f_s. It is the *Larmor* or precession frequency. It is (also) a classical thing: if we think of the electron as a tiny magnet with a magnetic moment that is proportional to its angular momentum, then it should, effectively, *precess* in a magnetic field.

The analysis of precession is quite straightforward. The geometry of the situation is shown below and we may refer to (almost) any standard physics textbook for the derivation.[6]



$$\omega_p = \frac{\mu}{\mathrm{J}} \cdot \mathrm{B} \Leftrightarrow \frac{\mu}{\mathrm{J}} = \frac{\omega_p}{\mathrm{B}}$$

It is tempting to use the equality above and write this as:

---

[5] J is the symbol which Feynman uses. In many articles and textbooks, one will read L instead of J. Note that the symbols may be confusing: I is a current, but *I* is the moment of inertia. It is equal to m·r^2 for a rotating mass.

[6] We like the intuitive – but precise – explanation in Feynman's *Lectures* (II-34-3), from which we also copied the illustration.

$$\omega_p = \frac{\mathrm{q}}{2\mathrm{m}} \cdot \mathrm{B}$$

However, we should *not* do this. The precession causes the electron to wobble: its plane of rotation – and, hence, the axis of the angular momentum (and the magnetic moment) – is no longer fixed. This wobbling motion changes the orbital and, therefore, we can no longer trust the values we have used in our formulas for the angular momentum and the magnetic moment. There is, therefore, nothing anomalous about the anomalous magnetic moment. In fact, we should not wonder why it is not zero, but – as we will argue – we should wonder why it is so *nearly* zero.

Let us continue our analysis. It is, in fact, a bit weird to associate a gyromagnetic ratio with this motion, but that is what the physicists doing these experiments do. We will denote this g-factor by g_p:

$$g_p = \frac{2\mathrm{m}}{\mathrm{q}}\frac{\mu}{\mathrm{J}} = \frac{2\mathrm{m}}{\mathrm{q}} \cdot \frac{\omega_p}{\mathrm{B}}$$

Hence, we can write the following *tautology*:

$$\omega_p = g_p \frac{\mathrm{q}}{2\mathrm{m}} \cdot \mathrm{B} = \frac{2\mathrm{m}}{\mathrm{q}} \cdot \frac{\omega_p}{\mathrm{B}} \cdot \frac{\mathrm{q}}{2\mathrm{m}} \cdot \mathrm{B} = \frac{\mu}{\mathrm{J}} \cdot \mathrm{B}$$

You can verify that this is nothing but a tautology by writing it all out:

$$\omega_p = g_p \frac{\mathrm{q}}{2\mathrm{m}} \cdot \mathrm{B} = \frac{2\mathrm{m}}{\mathrm{q}} \cdot \frac{\omega_p}{\mathrm{B}} \cdot \frac{\mathrm{q}}{2\mathrm{m}} \cdot \mathrm{B} = \omega_p$$

We can, of course, measure the frequency in cycles per second (as opposed to radians per second):

$$f_p = \frac{\omega_p}{2\pi} = g_p \frac{\mathrm{q} \cdot \mathrm{B}}{2\pi \cdot 2\mathrm{m}}$$

Hence, we get the following expression for the so-called anomalous magnetic moment of an electron a_e:

$$a_e = \frac{f_p - f_c}{f_c} = \frac{g_p \frac{\mathrm{q} \cdot \mathrm{B}}{2\pi \cdot 2\mathrm{m}} - g_c \frac{\mathrm{q} \cdot \mathrm{B}}{2\pi \cdot \mathrm{m}}}{g_c \frac{\mathrm{q} \cdot \mathrm{B}}{2\pi \cdot \mathrm{m}}} = \frac{1}{2}\frac{g_p}{g_c} - 1$$

Hence, the so-called anomalous magnetic moment of an electron is nothing but the ratio of two mathematical factors – definitions, basically – which we can express in terms of *actual* frequencies:

$$g_c = f_c \frac{2\pi \cdot \mathrm{m}}{\mathrm{q} \cdot \mathrm{B}}$$

$$g_p = f_p \frac{2\pi \cdot 2\mathrm{m}}{\mathrm{q} \cdot \mathrm{B}}$$

Our formula for a_e now becomes:

$$a_e = \frac{1}{2}\frac{g_p}{g_c} - 1 = \frac{1}{2}\frac{f_p \frac{2\pi \cdot 2\mathrm{m}}{\mathrm{q} \cdot \mathrm{B}}}{f_c \frac{2\pi \cdot \mathrm{m}}{\mathrm{q} \cdot \mathrm{B}}} - 1 = \frac{f_p}{f_c} - 1 = \frac{f_p - f_c}{f_c}$$

Of course, if we use the μ/J = 2m/q equality, then the f_p/f_c ratio will be equal to ½, and a_e will not be zero but −1/2.

$$\frac{f_p}{f_c} - 1 = \frac{\frac{\mathrm{q} \cdot \mathrm{B}}{2\pi \cdot 2\mathrm{m}}}{\frac{\mathrm{q} \cdot \mathrm{B}}{2\pi \cdot \mathrm{m}}} - 1 = \frac{1}{2} - 1 = -1/2$$

However, as mentioned above, we should not do that. The precession causes the magnetic moment and the angular momentum to wobble. Hence, there is nothing anomalous about the anomalous magnetic moment. We should not wonder why its value is not zero. We should wonder why it is so *nearly* zero.

References

All references are given in the footnotes.