Noname manuscript No.
(will be inserted by the editor)

Fermat's Last Theorem implies A Proof of The ABC Conjecture

Abdelmajid Ben Hadj Salem

Received: date / Accepted: date

Abstract In this paper, we use the Fermat's Last Theorem (FLT) to give a proof of the ABC conjecture. We suppose that FLT is false $\Longrightarrow$ we arrive that the ABC conjecture is false. Then taking the negation of the last statement, we obtain: ABC conjecture is true $\Longrightarrow$ FLT is true. But, as FLT is true, then we deduce that the ABC conjecture is true.

Keywords Prime numbers · Fermat's Last Theorem · Diophantine equations.

Mathematics Subject Classification (2010) 11AXX · 11D41

To the memory of my Father who taught me arithmetic.

1 *Introduction and notations*

Let a a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph OEsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

Abdelmajid Ben Hadj Salem
6, Rue du Nil, Cité Soliman Er-Riadh
8020 Soliman
Tunisia
E-mail: abenhadjsalem@gmail.com

Conjecture 1 (***ABC*** **Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{3}$$

where K is a constant depending only of ϵ.

This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. I try here to give a simple proof that can be understood by undergraduate students.
Our proof will use the Fermat's Last Theorem approved by Andrew John Wiles in 1993 [3].

We recall the Fermat's Last Theorem:

Theorem 1 *The equation :*

$$x^n + y^n = z^n \tag{4}$$

has no solutions with x, y, z all nonzero, relatively prime integers with $n > 2$ *a positive integer.*

The negation of the last theorem is:
It exists A, B, C *relatively prime integers and* $n > 2$ *a positive integer so that :*

$$A^n + B^n = C^n \tag{5}$$

2 *Methodology of the proof*

We denote :

$$\text{A: Fermat's Last Theorem} \tag{6}$$
$$\text{B: } ABC \text{ Conjecture} \tag{7}$$

and we use the following property (the contrapositive law, [4]) :

$$\boxed{A(False) \Longrightarrow B(False)} \Longleftrightarrow \boxed{B(True) \Longrightarrow A(True)} \tag{8}$$

From the right equivalent expression in the box above, as A (FLT) is true, then B (ABC Conjecture) is true.

3 *Proof of the conjecture* (1)

We suppose that FLT is false, then it exists A, B, C positive coprime integers and m a positive integer > 2 such:

$$A^m + B^m = C^m \tag{9}$$

the integers A, B, C, m are supposed large integers. We consider in the following that $A > B$. Now, we use the ABC conjecture for equation (9). We choose the value of $\epsilon \approx 0.001$, then it exists the constant $K(\epsilon) > 0$, we want to find if :

$$C^m \stackrel{?}{<} K(\epsilon) rad(A^m.B^m.C^m)^{1+\epsilon}$$
$$C^m \stackrel{?}{<} K(\epsilon)\left(rad(A).rad(B).rad(C)\right)^{1+\epsilon} \tag{10}$$

But $rad(A) \leq A < C$, $rad(B) \leq B < C$ and $rad(C) \leq C$, then we write (10) as :

$$C^m \stackrel{?}{<} K(\epsilon)\left(rad(A).rad(B).rad(C)\right)^{1+\epsilon} \Longrightarrow C^m \stackrel{?}{<} K(\epsilon) C^{3.(1+\epsilon)} \tag{11}$$

3.1 *Case* $K(\epsilon) \leq 1$

In this case, we obtain:

$$C^m \stackrel{?}{<} C^{3.(1+\epsilon)} \tag{12}$$

As $\epsilon \ll 1 \Longrightarrow 3(1+\epsilon) \ll m$, then $C^m > K(\epsilon) rad(A^m.B^m.C^m)^{1+\epsilon}$ and the ABC conjecture is false. Using the right member of the property (8), we obtain:

$$\boxed{ABC \text{ Conjecture True} \Longrightarrow \text{ FLT True}} \tag{13}$$

But as FLT holds, hence ABC Conjecture is true.

3.2 *Case* $K(\epsilon) > 1$ and $C^m > K(\epsilon)$

In this case, Let $\epsilon \approx 0.001$ and we suppose that $K(\epsilon) > 1$. As FLT is supposed false, we consider that it exits a solution of (9) such that $C^m > K(\epsilon)$ with $C^m \gg_C K(\epsilon)$ that means $\exists \lambda$ a positive constant depending of C such $C^m = \lambda.K(\epsilon)$ and $\lambda \approx C^h$ with $(m-h) < \dfrac{m}{2}$. Then :

$$C^m \stackrel{?}{<} K(\epsilon) C^{3(1+\epsilon)} \tag{14}$$

The last equation can be written as :

$$\lambda \stackrel{?}{<} C^{3(1+\epsilon)} \tag{15}$$

The equation (15) indicates that we can write $\lambda \approx C^3 \Longrightarrow \dfrac{m}{2} < 3 \Longrightarrow m < 6$, then the contradiction with $6 \ll m$. Hence :

$$C^m > K(\epsilon) rad(A^m.B^m.C^m)^{1+\epsilon}$$

and the ABC conjecture is false. Using the right member of the property (8), we obtain:

$$\boxed{ABC\,\text{Conjecture True} \Longrightarrow\ \text{FLT True}} \tag{16}$$

But as FLT holds, hence ABC Conjecture is true.

3.3 *Case* $K(\epsilon) > 1$ and $C^m < K(\epsilon)$

We consider $\epsilon = 0.001$ and we suppose that $K(\epsilon) > 1$. As FLT is supposed false, we consider that it exits a unique solution of (9) such that $C^m < K(\epsilon)$:

$$C^m = A^m + B^m \tag{17}$$

We obtain that:

$$C^m < K(\epsilon) rad(A^m.B^m.C^m)^{1+\epsilon} \tag{18}$$

and the ABC conjecture is true for $C^m = A^m + B^m$, but there is a contradiction because the hypothesis of the beginning used for the proof is false, then this case is to reject.

The proof of the ABC conjecture is achieved.

4 *Conclusion*

In the mathematical literature, the ABC conjecture, assumed true, is used to approve the Fermat's Last Theorem, in our paper, we have given a proof that the ABC conjecture is true using the Fermat's Last Theorem. We can announce the important theorem:

Theorem 2 *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2018) For each* $\epsilon > 0$ *, there exists* $K(\epsilon) > 0$ *such that if* a, b, c *positive integers relatively prime with* $c = a + b$*, then :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{19}$$

where K *is a constant depending only of* ϵ*.*

References

1. M. Waldschmidt. On the abc Conjecture and some of its consequences presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, (2013).
2. A. J. Wiles. Modular elliptic curves and Fermat's Last Theorem. Annals of Mathematics, **141**, 443-551. (1995).
3. Klaus Kremmerz for Quanta Magazine. Titans of Mathematics Clash Over Epic Proof of ABC Conjecture. The Quanta Newsletter, 20 September 2018. www.quantamagazine.org. (2018).
4. Daniel W. Cunningham. A Logical Introduction to Proof. ISBN 978-1-4614-3630-0. Springer New York Heidelberg Dordrecht London. 356 pages. (2012).