Complex and Quaternion Optimization

Yuly Shipilevsky
Toronto, Ontario, Canada
yulysh2000@yahoo.ca

Abstract. We introduce and suggest to research a special class of optimization problems, wherein an objective function is a real-valued complex variables function under constraints, comprising complex-valued complex variables functions: "Complex Optimization" or "Complex Programming".

We demonstrate multiple examples to show a rich variety of problems, describing complex optimization as an optimization subclass as well as a Mixed-Real-Integer Complex Optimization.

Next, we introduce more general concept "Quaternionic Optimization" for optimization over quaternion subsets.

Keywords: optimization, target function, constraints, minimizers, complex plane, complex variables function, quaternion

1. Introduction

Its well-known that an optimization problem can be represented in the following way:

> given: a function f: $\mathbf{G} \to \mathbf{R}$ from some set $\mathbf{G}$ to the real numbers,
> sought: an element $x_0 \in \mathbf{G}$ such that $f(x_0) \leq f(x)$ for all $x \in \mathbf{G}$ ("minimization") or such that $f(x_0) \geq f(x)$ for all $x \in \mathbf{G}$ ("maximization").

Typically, $\mathbf{G}$ is some subset of the Euclidean space $\mathbf{R}^n$, specified by a set of constraints and the function f is called an objective function, target function.

The case, when $\mathbf{G}$ is some subset of two-dimensional complex plane and target function f: $\mathbf{C} \to \mathbf{R}$ is real-valued complex variable function is very poor investigated yet.

In [1], the mathematical formulation of the ptychographic phase retrieval problem is examined and the corresponding minimization of intensity Gaussian error metric(for the corresponding discrete Fourier operator and probes: scanning positions) over potentially complex vector is investigated (Least Squares Problem). Phase retrieval is the process of algorithmically finding solutions to the phase problem. In physics, the phase problem is a problem of loss of information concerning the phase that can occur when making a physical measurement.

In [2], methods to solve unconstrained nonlinear optimization problems of real-valued complex functions in several complex variables are developed in order to overcome the fact that due to Cauchy-Riemann conditions, the real-valued functions in complex variables are necessarily nonanalytic.

As we see, currently, optimization of real-valued complex variable functions over the complex plane is considered and investigated just for some specific cases, no general models, comprising wide variety of targets and constraints are considered and investigated yet.

The purpose of this paper is to introduce and describe wide variety of optimization problems of f: $\mathbf{C} \to \mathbf{R}$ and f: $\mathbf{C}^n \to \mathbf{R}$ target functions over subsets of $\mathbf{C}$ and $\mathbf{C}^n$, specified by the constraints, comprising complex functions of one and several complex variables, and, further, we introduce more general concept: "Quaternionic Optimization", when **G** is some quaternion subset.

2. Complex Optimization

Basic definitions, methods and algorithms for modern complex analysis is considered, e.g., in [3] and [4].

Complex analysis proves a powerful tool for solving wide variety of problems in fundamental science and engineering: the analysis of electrical circuits, hydro- and aerodynamics, and so on.

In [5], [6], [7], formulas in various engineering applications and science are considered: e.g,, for electrical impedance, electromechanical circuits, electromagnetic fields, hydro- and aerodynamics. Imposing the corresponding constraints on complex input parameters we could obtain various complex optimization problems.

Let $| z |$ be the absolute value of a complex number $z = \mathrm{Re}(z) + \mathrm{Im}(z)\mathbf{i} = a + \mathbf{i}b$, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $\mathbf{i}^2 = -1$ and arg(z) the argument of z: the principal value. (See, e.g., [3], [4]).

Let us introduce and demonstrate various optimization problems, defined in terms of complex numbers and functions, contained in various targets and constraints.

Example 2.1.

cop1 = { minimize | z | subject to | z | ≥ 1 }, argmin(cp1) = {z: | z | = 1}.

Example 2. 2.

cop2 = { minimize − Im(z) subject to | z | ≤ 1 }, argmin(cp2) = **i**.

Example 2.3.

cop3 = { minimize Re(z) subject to | z | ≤ 1 }, argmin(cp3) = -1.

Example 2.4.

cop4 = { maximize | z | subject to 0 ≤ Re(z) ≤ 1, 0 ≤ Im(z) ≤ 1 },
argmax(cp4) = 1 + **i**.

Example 2.5.

cop5 = { maximize Re(z) + Im(z) subject to 0 ≤ Re(z) ≤ 1, 0 ≤ Im(z) ≤ 1 },
argmax(cp5) = 1 + **i**.

Example 2.6.

cop6 = { maximize Re(z) + Im(z) subject to 0 ≤ Re(z) ≤ 1, 0 ≤ Im(z) ≤ 1,
arg(z) = 0 }, argmax(cp6) = 1.

Example 2.7.

cop7 = { maximize Im(z) subject to 0 ≤ Re(z) ≤ 1, 0 ≤ Im(z) ≤ 1,
Im(z) ≤ Re(z) }, argmax(cp7) = 1 + **i**.

Example 2.8.

cop8 = { maximize | z | subject to Im(z) ≥ Re^2(z), Re(z) ≥ Im^2(z) },
argmax(cp8) = 1 + **i**.

Example 2.9. Polynomial Complex Optimization.

cop9 = { maximize $| c_n z^n + \ldots + c_1 z |$ subject to

$$
\begin{aligned}
&| a_{1n}z^n + \ldots + a_{11}z | \le b_1,\\
&\ \ \ldots \qquad\quad \ldots \qquad\quad \ldots\\
&| a_{mn}z^n + \ldots + a_{m1}z | \le b_m,
\end{aligned}
$$

$z \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$, $c_j \in \mathbf{C}$,

$1 \le i \le m$, $1 \le j \le n$, $n \in \mathbf{N}$, $m \in \mathbf{N}$ }.

(More sophisticated examples would contain rational meromorphic complex functions).

Example 2.10. Several complex variables.

cop10 = { maximize $| z_1 + z_2 |$ subject to $| z_1 | \le 1$, $| z_2 | \le 1$ }.

Example 2.11.

cop11 = { maximize $| z_1 + z_2 |$ subject to $| z_1 | \le 1$, $| z_2 | \le 1$, $\arg(z_1 z_2) \le \pi/4$ }.

Example 2.12. Linear Complex Optimization.

cop12a = { maximize $| c_1 z_1 + \ldots + c_n z_n |$ subject to

$$
\begin{aligned}
&| a_{11}z_1 + \ldots + a_{1n}z_n | \le b_1,\\
&\ \ \ldots \qquad\quad \ldots \qquad\quad \ldots\\
&| a_{m1}z_1 + \ldots + a_{mn}z_n | \le b_m,
\end{aligned}
$$

$z_j \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$, $c_j \in \mathbf{C}$,

$1 \le i \le m$, $1 \le j \le n$, $n \in \mathbf{N}$, $m \in \mathbf{N}$ }.

cop12b = { maximize $| c_1 z_1 + \ldots + c_n z_n |$ subject to

$$
\begin{aligned}
&a_{11}z_1 + \ldots + a_{1n}z_n = b_1,\\
&\ldots \qquad\quad \ldots \qquad\quad \ldots\\
&a_{m1}z_1 + \ldots + a_{mn}z_n = b_m,
\end{aligned}
$$

$z_j \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{C}$, $c_j \in \mathbf{C}$,

$(A\mathbf{z} = b)$,

$1 \le i \le m, 1 \le j \le n, n \in \mathbf{N}, m \in \mathbf{N}$ }.

Example 2.13.

cop13 = **{** maximize $| z_1 + ... + z_n |$ subject to

$$\mathrm{Re}(a_{11}z_1 + ... + a_{1n}z_n) \le b_1,$$
$$... \quad ... \quad ...$$
$$\mathrm{Re}(a_{m1}z_1 + ... + a_{mn}z_n) \le b_m,$$

$$\mathrm{Im}(a_{11}z_1 + ... + a_{1n}z_n) \le c_1,$$
$$... \quad ... \quad ...$$
$$\mathrm{Im}(a_{m1}z_1 + ... + a_{mn}z_n) \le c_m,$$

$z_j \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$, $c_i \in \mathbf{R}$,

$1 \le i \le m, 1 \le j \le n, n \in \mathbf{N}, m \in \mathbf{N}$ }.

Example 2.14.

cop14 = **{** maximize $| z_1 + ... + z_n |$ subject to

$$\arg(a_{11}z_1 + ... + a_{1n}z_n) \le b_1,$$
$$... \quad ... \quad ...$$
$$\arg(a_{m1}z_1 + ... + a_{mn}z_n) \le b_m,$$

$$\mathrm{Im}(a_{11}z_1 + ... + a_{1n}z_n) \le c_1,$$
$$... \quad ... \quad ...$$
$$\mathrm{Im}(a_{m1}z_1 + ... + a_{mn}z_n) \le c_m,$$

$z_j \in \mathbf{C}$, $a_{ij} \in \mathbf{C}$, $b_i \in \mathbf{R}$, $c_i \in \mathbf{R}$,

$1 \le i \le m, 1 \le j \le n, n \in \mathbf{N}, m \in \mathbf{N}$ }.

Example 2.15.

cop15 = { maximize $\arg(z_1 \ldots z_n)$ subject to

$$\mathrm{Re}(a_{11}z_1 + \ldots + a_{1n}z_n) \le b_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\mathrm{Re}(a_{m1}z_1 + \ldots + a_{mn}z_n) \le b_m,$$

$$\mathrm{Im}(a_{11}z_1 + \ldots + a_{1n}z_n) \le c_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\mathrm{Im}(a_{m1}z_1 + \ldots + a_{mn}z_n) \le c_m,$$

$$\arg(z_j) \le d_j,$$

$$z_j \in \mathbf{C},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},\ c_i \in \mathbf{R},\ d_j \in \mathbf{R},$$

$$1 \le i \le m,\ 1 \le j \le n,\ n \in \mathbf{N},\ m \in \mathbf{N}\ \}.$$

Example 2.16. Quadratic Complex Optimization.

cop16 = { maximize $| z_1^2 + \ldots + z_n^2 - \mathbf{i}z_1z_2 |$ subject to

$$| a_{11}z_1 + \ldots + a_{1n}z_n | \le b_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$| a_{m1}z_1 + \ldots + a_{mn}z_n | \le b_m,$$

$$z_j \in \mathbf{C},\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},$$

$$1 \le i \le m,\ 1 \le j \le n,\ n \in \mathbf{N},\ m \in \mathbf{N}\ \}.$$

Example 2.17. Non-Linear Complex Optimization.

cop17 = { maximize $| e^z + \sin(\pi z) |$ subject to

$$| \cos(\pi z) | \le a,\ 0 \le \mathrm{Re}(z) \le 1,\ 0 \le \mathrm{Im}(z) \le 1,$$

$$z \in \mathbf{C},\ a \in \mathbf{R}\ \}.$$

Example 2.18. Similar to the very well known in Mathematical Optimization Integer Optimization(see, e.g., [8]): optimization over integer points, we introduce here Complex Optimization over subsets of the Gaussian Integers ($\mathbf{C} \cap \mathbf{Z}^2$), **-** Integer Complex Optimization(ICOP).

cop18 = **{** maximize $| z_1^4 + \ldots + z_n^4 |$ subject to

$$b_1 \le | a_{11}z_1 + \ldots + a_{1n}z_n | \le c_1,$$
$$\ldots \quad \ldots \quad \ldots$$
$$b_m \le | a_{m1}z_1 + \ldots + a_{mn}z_n | \le c_m,$$

$$z_j \in \mathbf{C} \cap \mathbf{Z}^2,\ a_{ij} \in \mathbf{C},\ b_i \in \mathbf{R},\ c_i \in \mathbf{R},$$

$$1 \le i \le m,\ 1 \le j \le n,\ n \in \mathbf{N},\ m \in \mathbf{N}\ \}.$$

Example 2.19. Mixed-Real Complex Optimization (MRCOP).

cop19 = **{** maximize $| \mathbf{i}z^3 | - x^2 + y^3$ subject to

$$| z | \le a,\ \ b \le x \le c,\ d \le y \le e,$$

$$z \in \mathbf{C},\ x \in \mathbf{R},\ y \in \mathbf{R},\ a \in \mathbf{R},\ b \in \mathbf{R},\ c \in \mathbf{R},\ d \in \mathbf{R},\ e \in \mathbf{R}.$$

Example 2.20. Mixed-Real-Integer Complex Optimization (MRICOP).

cop20 = **{** minimize $| \mathbf{i}z_1^4 + z_2^2 | - x^2 + y^3t^2$ subject to

$$xy \ge N,$$

$$a_1 \le | z_1 | \le b_1,$$

$$a_2 \le | z_2 | \le b_2,$$

$$a_3 \le x \le b_3,$$

$$a_4 \le y \le b_4,$$

$$a_5 \le t \le b_5,$$

$z_1 \in \mathbf{C}$, $z_2 \in \mathbf{C} \cap \mathbf{Z}^2$,

$x \in \mathbf{Z}$, , $y \in \mathbf{Z}$, $t \in \mathbf{R}$,

$a_i \in \mathbf{R}$, $b_i \in \mathbf{R}$, $N \in \mathbf{N}$, $a_i > 0$,

$1 \le i \le 5$.

Note that in addition, each such example may comprise complex conjugations as well.

3. Quaternionic Optimization

Quaternions are generally represented in the form: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$, $d \in \mathbf{R}$, and **i**, **j** and **k** are the fundamental quaternion units and are a number system that extends the complex numbers(see, e.g., [9], [10]). Quaternions find uses in both pure and applied mathematics: in three-dimensional computer graphics, computer vision, robotics, control theory, signal processing, attitude control, physics, bioinformatics, molecular dynamics, computer simulations, orbital mechanics, crystallographic texture analysis. In quantum mechanics, the spin of an electron and other matter particles can be described using quaternions. In 1999 is was shown that Einstein equations of general relativity could be formulated using quaternions.

The set of all quaternions **H** is a normed algebra, where the norm is multiplicative: $\| pq \| = \| p \| \| q \|$, $p \in \mathbf{H}$, $q \in \mathbf{H}$, $\| q \|^2 = a^2 + b^2 + c^2 + d^2$.

This norm makes it possible to define the distance $d(p, q) = \|p - q\|$, which makes **H** into a metric space.

Let us introduce Quaternionic Optimization - optimization of real-valued quaternionic functions over quaternionic subsets.

Example 3.1. Linear Quaternionic Optimization.

qop1a = { maximize $\| c_1q_1 + ... + c_nq_n \|$ subject to

$\| a_{11}q_1 + ... + a_{1n}q_n \| \le b_1$,

... ...

$\| a_{m1}q_1 + ... + a_{mn}q_n \| \le b_m$,

$q_j \in \mathbf{H}$, $a_{ij} \in \mathbf{H}$, $b_i \in \mathbf{R}$, $c_j \in \mathbf{H}$,

$1 \le i \le m, 1 \le j \le n, n \in \mathbf{N}, m \in \mathbf{N}$ }.

qop1b = { maximize $\| c_1q_1 + ... + c_nq_n \|$ subject to

$a_{11}q_1 + ... + a_{1n}q_n = b_1,$
$\ldots \qquad \ldots \qquad \ldots$
$a_{m1}q_1 + ... + a_{mn}q_n = b_m,$

$q_j \in \mathbf{H}, a_{ij} \in \mathbf{H}, b_i \in \mathbf{H}, c_j \in \mathbf{H},$

$(A\mathbf{q} = b),$

$1 \le i \le m, 1 \le j \le n, n \in \mathbf{N}, m \in \mathbf{N}$ }.

Example 3.2. Polynomial Quaternionic Optimization.

qop2 = { maximize $\| c_nq^n + ... + c_1q \|$ subject to

$\| a_{1n}q^n + ... + a_{11}q \| \le b_1,$
$\ldots \qquad \ldots \qquad \ldots$
$\| a_{mn}q^n + ... + a_{m1}q \| \le b_m,$

$q \in \mathbf{H}, a_{ij} \in \mathbf{H}, b_i \in \mathbf{R}, c_j \in \mathbf{H},$

$1 \le i \le m, 1 \le j \le n, n \in \mathbf{N}, m \in \mathbf{N}$ }.

Example 3.3. Mixed-Complex-Real-Integer Quaternionic Optimization (MCRIQOP). Similar to Gaussian Integers in Complex Analysis, let us introduce a set of integer points for quaternions: $\mathbf{H} \cap \mathbf{Z}^4$.

qop3 = { minimize $\| p - q \| + | \mathbf{i}z_1^4 + z_2^2 | - x^2 + y^3t^2$ subject to

$xy \ge N,$

$a_1 \le \| p \| \le b_1,$

$a_2 \le \| q \| \le b_2,$

$a_3 \leq |z_1| \leq b_3$,

$a_4 \leq |z_2| \leq b_4$,

$a_5 \leq x \leq b_5$,

$a_6 \leq y \leq b_6$,

$a_7 \leq t \leq b_7$,

$p \in \mathbf{H}, q \in \mathbf{H} \cap \mathbf{Z}^4$,

$z_1 \in \mathbf{C}, z_2 \in \mathbf{C} \cap \mathbf{Z}^2$,

$x \in \mathbf{Z},, y \in \mathbf{Z}, t \in \mathbf{R}$,

$a_i \in \mathbf{R}, b_i \in \mathbf{R}, N \in \mathbf{N}, a_i > 0$,

$1 \leq i \leq 7$.

4. Open Problems

Despite such optimization problems actually could be translated and considered in terms of optimization problems over the Euclidean space, it may be not always so "easy" task(complexity problems, etc.).

That is why, it would be preferable to develop specific, "direct" methods for Complex and Quaternionic Optimization problems using Complex and Quaternionic Analysis.

The corresponding complexity evaluations for the Complex and Quaternionic Optimization problems would be developed as well: for example in binary encoded length of the coefficients(see, e.g., [8], [11]) and, in particular, finding conditions for the polynomial-time optimization.

Complex and Quaternionic Optimization ideas may be useful for similar approaches in other subfields of the Optimization Theory, e.g., in Optimal Control Theory.

5. Conclusions

We described a rich variety of optimization problems as classes of optimization problems, comprising complex numbers and complex functions in their targets and constraints: "Complex Optimization" and quaternion variables and functions: "Quaternionic Optimization".

References

1. Qian, J., Yang, C., Schirotzek, A., Maia, F. S., Marchesini, S.: Efficient algorithms for ptychographic phase retrieval. Inverse Problems and Applications, Contemporary Mathematics 625, 261–280, (2014).

2. Sorber, L., Van Barel, M., Lathauwer, L. De.: Unconstrained optimization of real functions in complex variables, SIAM Journal on Optimization, 22(3) , 879–898, (2012).

3. Scheidemann, V.: Introduction to complex analysis in several variables, Birkhauser, (2005).

4. Shaw, W. T.: Complex Analysis with Mathematica, Cambridge, (2006).

5. Campos, L. M. B. C.: Complex Analysis with Applications to Flows and Fields, CRC Press, (2011).

6. James, G. (ed): Modern Engineering Mathematics, Trans-Atlantic Pubns Inc., (2015).

7. Kreyszig, E. (ed): Advanced Engineering Mathematics, John Wiley & Sons Inc., (2011).

8. Hemmecke, R., Koppe, M., Lee, J., Weismantel, R.: Nonlinear Integer Programming, in: Junger, M., Liebling, T., Naddef, D., Pulleyblank, W., Reinelt, W., Rinaldi, G., Wolsey, L. (eds.): 50 Years of Integer Programming 1958-2008: The Early Years and State-of-the-Art Surveys, pp. 561-618, Springer-Verlag, Berlin, (2010).

9. Hamilton, W. R. (ed): Elements of Quaternions, Chelsea Publishing Company, (1969).

10. Gürlebeck, K., Sprößig, W.: Quaternionic analysis and elliptic boundary value problems, Birkhäuser, Basel, (1990).

11. Cormen, T., Leiserson, C., Rivest, R., Stein, C. (ed): Introduction To Algorithms, The MIT Press, Cambridge, (2009).

12. Floudas, C. A., Pardalos, P. M. (ed): Encyclopedia of Optimization, Springer, New York, (2009).