Divergence measure of intuitionistic fuzzy sets

Fuyuan Xiao[a,*]

[a]*School of Computer and Information Science, Southwest University, Chongqing, 400715, China*

Abstract

As a generation of fuzzy sets, the intuitionistic fuzzy sets (IFSs) have more powerful ability to represent and deal with the uncertainty of information. The distance measure between the IFSs is still an open question. In this paper, we propose a new distance measure between the IFSs on the basis of the Jensen–Shannon divergence. The new distance measure of IFSs not only can satisfy the axiomatic definition of distance measure, but also can discriminate the diference between the IFSs more better. As a result, the new distance measure can generate more reasonable results.

Keywords: Intuitionistic fuzzy sets, Distance measure, Jensen–Shannon divergence

---

1. Introduction

In order to deal with the imprecision and uncertainty of information, various efficient methodologies have been developed in the decision-making theory [1–3]. They mainly consist of the modified fuzzy sets theory [4, 5], evidence theory [6,

*Corresponding author.
Email address: xiaofuyaunswu@hotmail.com (Fuyuan Xiao)

7], etc. As a generation of fuzzy set, the intuitionistic fuzzy sets (IFSs) presented by Atanassov [8] could handle the uncertainty of information more accurately. Hence, the IFS theory has been widely investigated and utilized in a variety of fields [9, 10].

The distance measure or the similarity measure of IFSs, as an important math tool for decision-making, has already caught great attention of researcher in the past few years. Up to now, various kinds of distance measures or similarity measures have been exploited for the IFSs. In this paper, a new distance measure between the IFSs is defined from divergence perspective. The proposed distance measure is based on the Jensen–Shannon divergence, which considers the divergence between two IFSs in terms of the three dimensional representation of IFSs, i.e., the membership function, the non-membership function and the hesitation function. In this study, it has been proven that the new distance measure between the IFSs satisfy the properties of the axiomatic definition of distance measure. Furthermore, numerical examples reveal that the new distance measure can better distinguish the IFSs and generate reasonable results.

The remainder of this paper is structured as follows. Section 2 shortly introduces the preliminaries of this paper. In Section 3, a new distance measure of IFSs is defined and investigated. Section 4 illustrates the proposed distance measure. Finally, Section 5 provides a conclusion.

2. Preliminaries

2.1. Intuitionistic fuzzy sets

Definition 2.1 *[8] Let X be a finite universe of discourse. An intuitionistic fuzzy set (IFS) E in the finite universe of discourse X is defined by*

$$E = \{\langle x, \mu_E(x), \nu_E(x)\rangle | x \in X\}, \tag{1}$$

in which

$$\mu_E(x) : X \to [0, 1] \quad and \quad \nu_E(x) : X \to [0, 1] \tag{2}$$

with the condition

$$0 \leq \mu_E(x) + \nu_E(x) \leq 1 \quad \forall x \in X. \tag{3}$$

The $\mu_E(x)$ and $\nu_E(x)$ represent the grade of membership and non-membership of x to E, respectively.

For the IFS A in X, the intuitionistic index of x to E is defined by

$$\pi_E(x) = 1 - \mu_E(x) - \nu_E(x), \tag{4}$$

which is a hesitancy grade of x to E.

2.2. Jensen–Shannon divergence measure

For the Jensen–Shannon divergence measure, its square root is a true metric in the space of probability distributions [11]. It was regarded as an useful distance measure which was applied in many fields [12].

Definition 2.2 *[11] Let E and F be two probability distributions of a discrete random variable U, where $E = \{e_1, e_2, \ldots, e_n\}$ and $F = \{f_1, f_2, \ldots, f_n\}$. The Jensen-Shannon divergence between E and F is defined by*

$$JS(E, F) = \frac{1}{2}\left[KL\left(E, \frac{E+F}{2}\right) + KL\left(F, \frac{E+F}{2}\right)\right], \tag{5}$$

where $KL(E,F)=\sum_i e_i \log \frac{e_i}{f_i}$ $(1\le i\le n)$ *is the Kullback-Leibler divergence and* $\sum_i e_i=\sum_i f_i=1$.

$JS(E,F)$ *can also be formed as*

$$\begin{aligned} JS(E,F) &= H\left(\frac{E+F}{2}\right)-\frac{1}{2}H(E)-\frac{1}{2}H(F), \\ &= \frac{1}{2}\left[\sum_i e_i \log\left(\frac{2e_i}{e_i+f_i}\right)+\sum_i f_i \log\left(\frac{2f_i}{e_i+f_i}\right)\right], \end{aligned} \tag{6}$$

where $H(E)=-\sum_i e_i \log e_i$ *and* $H(F)=-\sum_i f_i \log f_i$ $(1\le i\le n)$ *are the Shannon entropy.*

The square root of Jensen-Shannon divergence is defined by

$$SR_{JS}=\sqrt{JS(E,F)}. \tag{7}$$

3. A new distance measure of IFSs

Definition 3.1 *Given a finite universe of discourse* X*, and two intuitionistic fuzzy sets* $P=\{\langle x,\mu_P(x),\nu_P(x)\rangle | x\in X\}$ *and* $Q=\{\langle x,\mu_Q(x),\nu_Q(x)\rangle | x\in X\}$, *where* $\pi_P(x)=1-\mu_P(x)-\nu_P(x)$ *and* $\pi_Q(x)=1-\mu_Q(x)-\nu_Q(x)$ *are the hesitancy grades of* x *to* P *and* Q*, respectively. The intuitionistic fuzzy divergence measure, denoted as* $JS_{IFS}(P,Q)$ *between two IFSs* P *and* Q *is defined by*

$$JS_{IFS}(P,Q)=\frac{1}{2}\left[KL\left(P,\frac{P+Q}{2}\right)+KL\left(Q,\frac{P+Q}{2}\right)\right], \tag{8}$$

with

$$KL(P,Q)=\mu_P(x)\log\frac{\mu_P(x)}{\mu_Q(x)}+\nu_P(x)\log\frac{\nu_P(x)}{\nu_Q(x)}+\pi_P(x)\log\frac{\pi_P(x)}{\pi_Q(x)}, \tag{9}$$

where $KL(P,Q)$ *is the Kullback-Leibler divergence.*

$JS_{IFS}(P,Q)$ can also be expressed by the following formula

$$\begin{aligned}
JS_{IFS}(P,Q) = H\left(\frac{P+Q}{2}\right) - \frac{1}{2}H(P) - \frac{1}{2}H(Q),\\
= \frac{1}{2}\Bigg[\mu_P(x)\log\frac{2\mu_P(x)}{\mu_P(x)+\mu_Q(x)} + \mu_Q(x)\log\frac{2\mu_Q(x)}{\mu_P(x)+\mu_Q(x)}\\
+\nu_P(x)\log\frac{2\nu_P(x)}{\nu_P(x)+\nu_Q(x)} + \nu_Q(x)\log\frac{2\nu_Q(x)}{\nu_P(x)+\nu_Q(x)}\\
+\pi_P(x)\log\frac{2\pi_P(x)}{\pi_P(x)+\pi_Q(x)} + \pi_Q(x)\log\frac{2\pi_Q(x)}{\pi_P(x)+\pi_Q(x)}\Bigg],
\end{aligned} \tag{10}$$

with

$$H(P) = -\left(\mu_P(x)\log\mu_P(x) + \nu_P(x)\log\nu_P(x) + \pi_P(x)\log\pi_P(x)\right), \tag{11}$$

and

$$H(Q) = -\left(\mu_Q(x)\log\mu_Q(x) + \nu_Q(x)\log\nu_Q(x) + \pi_Q(x)\log\pi_Q(x)\right), \tag{12}$$

where $H(P)$ and $H(Q)$ are the Shannon entropy.

Then, we define a new distance measure for the IFSs in accordance with the intuitionistic fuzzy divergence.

Definition 3.2 *Let P and Q be two intuitionistic fuzzy sets in the finite universe of discourse X. A new distance measure for the IFSs, denoted as $d_\chi(P,Q)$ between the IFSs P and Q is defined by*

$$d_\chi(P,Q) = \sqrt{JS_{IFS}(P,Q)}. \tag{13}$$

The properties of the new distance measure for the IFSs are deduced as follows:

Property 1 *Let P, Q and K be three IFSs in X, then*

P1. $d_\chi(P,Q) = 0$ iff $P = Q$, for $P, Q \in X$, *P2.* $d_{\chi}(P,Q) = d_{\chi}(Q,P)$, *for* $P, Q \in X$,

P3. $d_{\chi}(K,P) + d_{\chi}(P,Q) \geq d_{\chi}(K,Q)$, *for* $K, P, Q \in X$,

P4. $0 \leq d_{\chi}(P,Q) \leq 1$, *for* $P, Q \in X$.

Definition 3.3 *Let P and Q be two IFSs in a finite universe of discourse $X = \{x_1, x_2, ..., x_n\}$, where $P = \{\langle x_i, \mu_P(x), \nu_P(x)\rangle | x_i \in X\}$ and $Q = \{\langle x_i, \mu_Q(x), \nu_Q(x)\rangle | x_i \in X\}$. The normalized d_{χ} distance measure between P and Q is defined by*

$$\begin{aligned} d_{\widetilde{\chi}}(P,Q) =& \frac{1}{n}\sum_{i=1}^{n} d_{\chi}(P,Q) \\ =& \frac{1}{n}\left[\frac{1}{2}\left(\mu_P(x_i)\log\frac{2\mu_P(x_i)}{\mu_P(x_i)+\mu_Q(x_i)} + \mu_Q(x_i)\log\frac{2\mu_Q(x_i)}{\mu_P(x_i)+\mu_Q(x_i)}\right.\right. \\ &+\nu_P(x_i)\log\frac{2\nu_P(x_i)}{\nu_P(x_i)+\nu_Q(x_i)} + \nu_Q(x_i)\log\frac{2\nu_Q(x_i)}{\nu_P(x_i)+\nu_Q(x_i)} \\ &\left.\left.+\pi_P(x_i)\log\frac{2\pi_P(x_i)}{\pi_P(x_i)+\pi_Q(x_i)} + \pi_Q(x_i)\log\frac{2\pi_Q(x_i)}{\pi_P(x_i)+\pi_Q(x_i)}\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{14}$$

4. Numerical example

Example 1 Assume there are three IFSs A, B and C in the universe of discourse X:

$$A = \{\langle x, 0.30, 0.20\rangle\ \langle x, 0.40, 0.30\rangle\};$$

$$B = \{\langle x, 0.30, 0.20\rangle\ \langle x, 0.40, 0.30\rangle\};$$

$$C = \{\langle x, 0.15, 0.25\rangle\ \langle x, 0.25, 0.35\rangle\}.$$

By Eq. (14), the distances between the IFSs A, B and C are measured as

$$d_{\widetilde{\chi}}(A,B) = 0.0000, \quad d_{\widetilde{\chi}}(B,A) = 0.0000;$$
$$d_{\widetilde{\chi}}(A,C) = 0.1463, \quad d_{\widetilde{\chi}}(C,A) = 0.1463;$$
$$d_{\widetilde{\chi}}(C,B) = 0.1463, \quad d_{\widetilde{\chi}}(B,C) = 0.1463.$$

We can see that $d_{\widetilde{\chi}}(A,B)$ is equal to zero, and $d_{\widetilde{\chi}}(A,C) = d_{\widetilde{\chi}}(B,C) = 0.1463$, since the IFS A is the same as the IFS B. Moreover, it can be also seen that $d_{\widetilde{\chi}}(A,B) = d_{\widetilde{\chi}}(B,A) = 0.0000$, $d_{\widetilde{\chi}}(A,C) = d_{\widetilde{\chi}}(C,A) = 0.1463$ and $d_{\widetilde{\chi}}(C,B) = d_{\widetilde{\chi}}(B,C) = 0.1463$.

5. Conclusion

In this paper, a new distance measure of IFSs was proposed to deal with the problem of decision-making. The main contribution of this study is to measure the difference between the IFSs by taking advantage of Jensen–Shannon divergence. The new method has the promising aspects in inference problem with IFSs.

Conflict of Interest

The author states that there are no conflicts of interest.

References

[1] A. Mardani, A. Jusoh, E. K. Zavadskas, Fuzzy multiple criteria decision-making techniques and applications–Two decades review from 1994 to 2014, Expert Systems with Applications 42 (8) (2015) 4126–4148.

[2] X. Deng, Analyzing the monotonicity of belief interval based uncertainty measures in belief function theory, International Journal of Intelligent Systems (2018) Published online, DOI: 10.1002/int.21999.

[3] W. Deng, X. Lu, Y. Deng, Evidential model validation under epistemic uncertainty, Mathematical Problems in Engineering (2018) DOI: 10.1155/2018/6789635.

[4] R. R. Yager, Fuzzy rule bases with generalized belief structure inputs, Engineering Applications of Artificial Intelligence 72 (2018) 93–98.

[5] H. Zheng, Y. Deng, Evaluation method based on fuzzy relations between Dempster–Shafer belief structure, International Journal of Intelligent Systems 33 (7) (2018) 1343–1363.

[6] X. Su, S. Mahadevan, W. Han, Y. Deng, Combining dependent bodies of evidence, Applied Intelligence 44 (2016) 634–644.

[7] W. Jiang, T. Yang, Y. Shou, Y. Tang, W. Hu, Improved evidential fuzzy c-means method, Journal of Systems Engineering and Electronics 29 (1) (2018) 187–195.

[8] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and Systems 20 (1) (1986) 87–96.

[9] L. Fei, H. Wang, L. Chen, Y. Deng, A new vector valued similarity measure for intuitionistic fuzzy sets based on OWA operators, Iranian Journal of Fuzzy Systems 15 (5) (2017) 31–49.

[10] R. R. Yager, OWA aggregation of intuitionistic fuzzy sets, International Journal of General Systems 38 (6) (2009) 617–641.

[11] J. Lin, Divergence measures based on the shannon entropy, IEEE Transactions on Information Theory 37 (1) (1991) 145–151.

[12] F. Xiao, Multi-sensor data fusion based on the belief divergence measure of evidences and the belief entropy, Information Fusion 46 (2019) (2019) 23–32.