A Complete Proof Of The ABC Conjecture

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Abstract: In this paper, supposing that Beal conjecture is true, we give a complete proof of the *ABC* conjecture. We consider that Beal conjecture is false $\Longrightarrow$ we arrive that the *ABC* conjecture is false. Then taking the negation of the last statement, we obtain: *ABC* conjecture is true $\Longrightarrow$ Beal conjecture is true. But, if the Beal conjecture is true, then we deduce that the *ABC* conjecture is true.

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To the memory of my Father who taught me arithmetic.

1. *Introduction and notations*

Let a a positive integer, $a=\prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a=\prod_i a_i^{\alpha_i}=rad(a).\prod_i a_i^{\alpha_i-1} \tag{1.1}$$

We note:

$$\mu_a=\prod_i a_i^{\alpha_i-1} \Longrightarrow a=\mu_a.rad(a) \tag{1.2}$$

The *ABC* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph OEsterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *ABC* conjecture is given above:

Conjecture 1.3. *(**ABC Conjecture**): For each $\varepsilon>0$, there exists $K(\varepsilon)>0$ such that if a,b,c positive integers relatively prime with $c=a+b$, then :*

$$c<K(\varepsilon).rad(abc)^{1+\varepsilon} \tag{1.4}$$

where K is a constant depending only of ε.

This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [3]. I try here to give a simple proof that can be understood by undergraduate students. Our proof will suppose that Beal conjecture is true. A paper giving the proof is under reviewing by the referees of Journal of European Mathematical Society ([2]).

We recall the Beal conjecture:

Conjecture 1.5. *Let $A,B,C,m,n,$ and l be positive integers with $m,n,l>2$. If:*

$$A^m+B^n=C^l \tag{1.6}$$

then $A,B,$ and C have a common factor.

2. *Methodology of the proof*

We note :

$$\text{A: Beal Conjecture} \tag{2.1}$$

$$\text{B: } ABC \text{ Conjecture} \tag{2.2}$$

and we use the following property :

$$\boxed{A(False) \Longrightarrow B(False)} \Longleftrightarrow \boxed{B(True) \Longrightarrow A(True)} \tag{2.3}$$

From the right equivalent expression in the box above, as A (the Beal Conjecture) is supposing true, then B (*ABC* Conjecture) is true.

3. *Proof of the conjecture* (1.5)

We suppose that Beal conjecture is false, then it exists A,B,C positive coprime integers and m,n,l positive integers all >2 such:

$$A^m + B^n = C^l \tag{3.1}$$

the integers A,B,C,m,n,l are supposed large integers. We consider in the following that $A^m > B^n$. Now, we use the ABC conjecture for equation (3.1). We choose the value of $\varepsilon \approx 0.001$, then it exists the constant $K(\varepsilon) > 0$, such:

$$C^l < K(\varepsilon) rad(A^m B^n C^l)^{1+\varepsilon}$$
$$C^l < K(\varepsilon)\left(rad(A).rad(B).rad(C)\right)^{1+\varepsilon} \tag{3.2}$$

But $rad(A) \leq A < C^{\frac{l}{m}}$, $rad(B) \leq B < C^{\frac{l}{n}}$ and $rad(C) \leq C$, then we write (3.2) as :

$$C^l < K(\varepsilon)\left(rad(A).rad(B).rad(C)\right)^{1+\varepsilon} \Longrightarrow C^l \overset{?}{<} K(\varepsilon) C^{\left(1+\frac{l}{m}+\frac{l}{n}\right).(1+\varepsilon)} \tag{3.3}$$

3.1 *Case* $K(\varepsilon) \leq 1$

In this case, we obtain:

$$C^l \overset{?}{<} C^{\left(1+\frac{l}{m}+\frac{l}{n}\right).(1+\varepsilon)} \tag{3.4}$$

As $\varepsilon \ll 1$, $(1+\varepsilon).\left(1+\dfrac{l}{m}+\dfrac{l}{n}\right) < l$, then $C^l > K(\varepsilon) rad(A^m B^n C^l)^{1+\varepsilon}$ and the *ABC* conjecture is false. Using the right member of the property (2.3), we obtain:

$$\boxed{ABC\,\text{Conjecture True} \Longrightarrow \text{ Beal Conjecture True}} \tag{3.5}$$

But as Beal Conjecture is supposed true, hence *ABC* Conjecture is true.

3.2 *Case* $K(\varepsilon) > 1$ **and** $C^l > K(\varepsilon)$

In this case, Let $\varepsilon \approx 0.001$ and we suppose that $K(\varepsilon) > 1$. As Beal conjecture is false, we consider that it exits a solution of (3.1) such that $C^l > K(\varepsilon)$ with $C^l \gg_C K(\varepsilon)$ that means that $\exists \lambda$ a positive constant depending of C such $C^l = \lambda.K(\varepsilon)$ and $\lambda \approx C^h$ with $(l-h) < \frac{l}{2}$. Then :

$$C^l \overset{?}{<} K(\varepsilon)\left(C^{1+\frac{l}{m}+\frac{l}{m}}\right)^{1+\varepsilon} \tag{3.6}$$

The last equation can be written as :

$$\lambda \overset{?}{<} \left(C^{1+\frac{l}{m}+\frac{l}{n}}\right)^{1+\varepsilon} \tag{3.7}$$

or:

$$\lambda \stackrel{?}{<} C^{\left(1+\frac{l}{m}+\frac{l}{n}\right).(1+\varepsilon)} \tag{3.8}$$

Let :

$$q = \left(1+\frac{l}{m}+\frac{l}{n}\right).(1+\varepsilon) \tag{3.9}$$

$$\varepsilon' = \frac{l}{m}+\frac{l}{n} \tag{3.10}$$

3.2.1 *Case* $m > l$ *and* $n > l$

In this case, $\varepsilon' < 2 \Longrightarrow q = (1+\varepsilon)(1+\varepsilon') \approx 2$, using (3.8), we arrive to $\lambda < C^2$ which is a contradiction with $(l-h) < l/2$, l is supposed a large integer, then *ABC* conjecture is false. Using (3.5), we deduce that the *ABC* conjecture is true.

3.2.2 *Case* $m < l$ *and* $n < l$

In this case, if $C > A \Rightarrow C^m > A^m > B^n \Rightarrow C^m > B^n \Rightarrow\Rightarrow C^m > C^l - A^m \Rightarrow A^m > C^l - C^m \Rightarrow A^m > C^m(C^{l-m}-1)$. As $l > m \Rightarrow C^{l-m}-1 > 1$, then $A^m > C^m \Longrightarrow A > C$ that is a contradiction with $C > A$. Hence $C < A$. We rewrite equations (3.2):

$$C^l < K(\varepsilon) rad(A^m B^n C^l)^{1+\varepsilon}$$
$$C^l < K(\varepsilon)\left(rad(A).rad(B).rad(C)\right)^{1+\varepsilon} \leq K(\varepsilon)\left(A.B.C\right)^{1+\varepsilon}$$
$$\Rightarrow C^l < K(\varepsilon)\left(A.B.C\right)^{1+\varepsilon} \tag{3.11}$$

Then:

$$C^l \stackrel{?}{<} K(\varepsilon)\left(A.B.C\right)^{1+\varepsilon} \tag{3.12}$$

As $B^m < A^n \Rightarrow B < A^{\frac{n}{m}}$ and $C < A$, then we obtain:

$$C^l \stackrel{?}{<} K(\varepsilon) A^{\left(2+\frac{n}{m}\right)(1+\varepsilon)} \tag{3.13}$$

If $m > n$, we have:

$$C^l \stackrel{?}{<} K(\varepsilon) A^2 \tag{3.14}$$

we arrive to $\lambda < A^2 \leq A^{m/2} \leq C^{l/2}$ which is a contradiction with $(l-h) < l/2$, l is supposed a large integer, then *ABC* conjecture is false. Using (3.5), we deduce that the *ABC* conjecture is true.

We suppose that $m < n$. If $B > A \Rightarrow B^n > A^n \Rightarrow B^n > A^n > A^m \Longrightarrow B^n > A^m$, it is a contradiction with $A^m > B^n$. Then $B < A$ and equation (3.12) becomes:

$$C^l \stackrel{?}{<} K(\varepsilon)\left(A.A.A\right)^{1+\varepsilon} \Longrightarrow C^l \stackrel{?}{<} K(\varepsilon) A^{3(1+\varepsilon)} \approx K(\varepsilon) A^3 \tag{3.15}$$

we arrive to $\lambda < A^3 \leq A^{m/2} \leq C^{l/2}$ which is a contradiction with $(l-h) < l/2$, l is supposed a large integer, then *ABC* conjecture is false. Using (3.5), we deduce that the *ABC* conjecture is true.

3.2.3 Case $m < l$ and $n > l$

If $C < A$, as $l < n \Rightarrow C^l < A^n \Rightarrow 0 < A^m < A^n - B^n$ then $\boxed{A > B}$. As $C^n > C^l > B^n \Rightarrow C^n > B^n \Rightarrow \boxed{C > B}$. So we obtain :

$$\boxed{B < C < A} \tag{3.16}$$

Then the equation (3.12) becomes:

$$C^l \stackrel{?}{<} K(\varepsilon)\,(A.B.C)^{1+\varepsilon} \Longrightarrow C^l \stackrel{?}{<} K(\varepsilon)\left(A.A^{l/n}.A\right)^{1+\varepsilon} \Rightarrow C^l \stackrel{?}{<} K(\varepsilon)A^{(2+l/n)(1+\varepsilon)} \approx K(\varepsilon)A^2 \tag{3.17}$$

we arrive to $\lambda < A^2 \leq A^{m/2} \leq C^{l/2}$ which is a contradiction with $(l-h) < l/2$, l is supposed a large integer, then ABC conjecture is false. Using (3.5), we deduce that the ABC conjecture is true.

If $A < C \Rightarrow A^l < C^l$ but $B^n < A^m \Rightarrow A^l < 2A^m \Rightarrow A^l < A^{m+1} \Rightarrow l < m+1$, as $m < l \Rightarrow m+1 \leq l < m+1$ that is a contradiction, then $\boxed{C < A}$ and this case is studied above.

3.2.4 Case $m > l$ and $n < l$

We have $n < l < m$. As $A^m < C^l \Rightarrow A < C^{\frac{l}{m}} < C \Rightarrow \boxed{A < C}$. As $2B^n < C^l \Rightarrow B < \dfrac{C^{\frac{l}{n}}}{2^{\frac{1}{n}}}$. The equation (3.12) becomes:

$$C^l \stackrel{?}{<} K(\varepsilon)\,(A.B.C)^{1+\varepsilon} \Longrightarrow C^l \stackrel{?}{<} K(\varepsilon)\left(C^{\frac{l}{m}}.\frac{C^{\frac{l}{n}}}{2^{\frac{1}{n}}}.C\right)^{1+\varepsilon}$$

$$\Rightarrow C^l \stackrel{?}{<} K(\varepsilon)2^{-\frac{1+\varepsilon}{n}}C^{(1+l/m+l/n)(1+\varepsilon)} < K(\varepsilon)C^{1+l/m+l/n} \approx K(\varepsilon)C^{1+l/n} \tag{3.18}$$

Then:

$$\lambda \approx C^{1+l/n} \tag{3.19}$$

As it supposed that $\lambda \approx C^h$ with $(l-h) < \frac{l}{2}$, we find that $l-h = l-1-l/n < l/2 \Rightarrow l - l/n \leq 1/2 \Rightarrow n \leq 2$ that is contradiction with $n \geq 3$, then the ABC conjecture is false. Using (3.5), we deduce that the ABC conjecture is true.

3.3 *Case* $K(\varepsilon) > 1$ and $C^l < K(\varepsilon)$

We consider $\varepsilon = 0.001$ and we suppose that $K(\varepsilon) > 1$. As Beal conjecture is false, we consider that it exits a unique solution of (3.1) such that $C^l < K(\varepsilon)$:

$$C^l = A^m + B^n \tag{3.20}$$

We obtain that:

$$C^l < K(\varepsilon)R(A^mB^nC^l)^{1+\varepsilon} \tag{3.21}$$

and the ABC conjecture is true for $C^l = A^m + B^n$, but there is a contradiction because the hypothesis of the beginning used for the proof is false, then this case is to reject.

The proof of the ABC conjecture is achieved

4. *Conclusion*

Supposing that Beal conjecture is true, we have given a proof that the *ABC* conjecture is true. We can announce the important theorem:

Theorem 1. *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2018) Let a,b,c positive integers relatively prime with $c = a+b$, then for each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that :*

$$c < K(\varepsilon).rad(abc)^{1+\varepsilon} \tag{4.1}$$

where $K(\varepsilon)$ depends only of ε.

References

[1] M. Waldschmidt. 2013. On the abc Conjecture and some of its consequences presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, 2013.

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[4] T. Gowers. 2008. The Princeton companion to mathematics. Princeton, ISBN 0691118809, 1027 pages.