On the distributional expansions of powered extremes from Maxwell distribution*

Jianwen Huang[a], Jianjun Wang[a], Zhongquan Tan[b], Jingyao Hou[a], Hao Pu[c]

[a]School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China

[b]Department of Statistics, Jiaxing University, Jiaxing, 314000, China

[c]School of Mathematics, Zunyi Normal College, Zunyi, 563002, China

Abstract. In this paper, asymptotic expansions of the distributions and densities of powered extremes for Maxwell samples are considered. The results show that the convergence speeds of normalized partial maxima relies on the powered index. Additionally, compared with previous result, the convergence rate of the distribution of powered extreme from Maxwell samples is faster than that of its extreme.

Keywords. Asymptotic expansion; density; Maxwell distribution; powered extreme.

AMS Classification: Primary 62E20, 60E05; Secondary 60F15, 60G15.

1 Introduction

In extreme value theory, researchers recently focus on investigating the quality of convergence of normalized $\max\{X_k, 1 \leq k \leq n\} := M_n$ of a sample. For the convergence rate of normalized M_n, general cases were discussed by Smith [1], Leadbetter et al. [2], Galambos [3] and de Haan and Resnick [4], and specific cases were considered by Hall [5, 6], Nair [7], Liao and Peng [8], Lin et al. [9, 10], Du and Chen [11, 12], and Huang et al. [13]. Hall [6] derived the asymptotics of distribution of normalized $|M_n|^t$, the powered extremes for given power index $t > 0$. Zhou and Ling [14] improved Hall' results and proved that the convergence speed of distributions and densities of extremes depends on the power index. Nair [7] established the asymptotic expansions of normalized maximum from normal samples. Liao et al. [15] and Jia et al. [16] generalized Nair's work to skew-normal distribution and general error distribution, respectively.

Since the Maxwell distribution was proposed by James Clerk Maxwell [17], a variety of applications of it in physics (in particular in statistical mechanics) have been found; see Shim and Gatignol [18], Tomer and Panwar [19] and Shim [20] and some statisticians and reliability engineers have investigated the statistical properties of it as well, see [13, 21–27].

The aim of this paper is to investigate the distributional tail representation of $|X|^t$ with X following Maxwell distribution and the limiting distribution of normalized $|M_n|^t$, and obtain asymptotic expansions of distribution and density of powered maximum from Maxwell distribution.

*Corresponding author. E-mail address: wjj@swu.edu.cn (J. Wang).

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables with marginal cumulative distribution function (cdf) F obeying the Maxwell distribution (abbreviated as $F \sim MD$), and as before let $M_n = \max\{X_i, 1 \leq i \leq n\}$ denote the partial maximum of $\{X_n, n \geq 1\}$. The probability density function (pdf) of the MD is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{x^2}{\sigma^3} \exp\left(-\frac{x^2}{\sigma^2}\right), \quad x > 0, \tag{1.1}$$

where $\sigma > 0$ is the scale parameter. Figure 1 presents the graph of pdf of Maxwell distribution. It shows that with the scale parameter increasing, the tail of pdf of MD becomes much heavier.

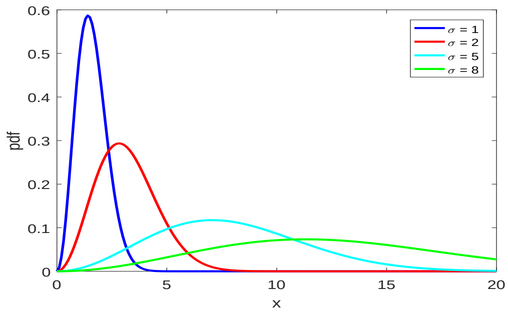

Figure 1: Probability density function of Maxwell distribution

Liu and Liu [21] showed that $F \in D(\Lambda)$, i.e., the max-domain of attraction of Gumbel extreme value distribution and the normalizing constants a_n and b_n can be given by

$$a_n = \sigma^2 b_n^{-1} \tag{1.2}$$

and

$$\sqrt{\frac{\pi}{2}} \frac{\sigma}{b_n} \exp\left(\frac{b_n^2}{2\sigma^2}\right) = n \tag{1.3}$$

such that

$$\lim_{n\to\infty} \mathbb{P}(M_n \leq a_n x + b_n) = \Lambda(x) = \exp\{-\exp(-x)\}. \tag{1.4}$$

The paper is constructed as follows. Section 2 presents auxiliary lemmas with proofs. The main results are given in Section 3. Section 4 provides the proofs of main results.

2 Auxiliary results

To prove the main results, the following auxiliary lemmas are needed.

Lemma 2.1. *Let $F(x)$ and $f(x)$ respectively represent the cdf and the pdf of MD with $\sigma > 0$, respectively. For large x, we have*

$$1 - F(x) = \sigma^2 x^{-1} f(x) \left[1 + \sigma^2 x^{-2} - \sigma^4 x^{-4} + 3\sigma^6 x^{-6} + O(x^{-8})\right]. \tag{2.1}$$

The proof of Lemma 2.1 is derived by integration by parts.

The following lemma gives the distributional tail representation of X^t with $X \sim MD$.

Lemma 2.2. *Suppose that $0 < t \neq 2$. Let $F_t(x)$ denote the cdf of X^t with $X \sim MD$. Then for large x, we get*

$$1 - F_t(x) = C_t(x) \exp\left\{ -\int_1^x \frac{g_t(u)}{\tilde{f}_t(u)} du \right\}, \tag{2.2}$$

where

$$C_t(x) \to \frac{2}{\sigma}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2\sigma^2}\right) \text{ as } x \to \infty,$$
$$g_t(x) = 1 - \sigma^2 x^{-2/t} \to 1 \text{ as } x \to \infty,$$

and

$$\tilde{f}_t(x) = \sigma^2 t x^{1-\frac{2}{t}} \text{ with } \tilde{f}_t'(x) \to 0 \text{ as } x \to \infty. \tag{2.3}$$

Proof. Combining with (2.1), we get

$$\begin{aligned} 1 - F_t(x) &= 2\frac{\sigma^2 f(x^{\frac{1}{t}})}{x^{\frac{1}{t}}} \left[1 + \sigma^2 x^{-\frac{2}{t}} - \sigma^4 x^{-\frac{4}{t}} + 3\sigma^6 x^{-\frac{6}{t}} + O(x^{-\frac{8}{t}})\right] \\ &= \frac{2}{\sigma}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{x^{\frac{2}{t}}}{2\sigma^2} + \frac{1}{t}\log x\right) \left[1 + \sigma^2 x^{-\frac{2}{t}} - \sigma^4 x^{-\frac{4}{t}} + 3\sigma^6 x^{-\frac{6}{t}} + O(x^{-\frac{8}{t}})\right] \\ &= C_t(x) \exp\left(-\int_1^x \frac{g_t(u)}{\tilde{f}_t(u)} du\right) \left[1 + \sigma^2 x^{-\frac{2}{t}} - \sigma^4 x^{-\frac{4}{t}} + 3\sigma^6 x^{-\frac{6}{t}} + O(x^{-\frac{8}{t}})\right] \end{aligned} \tag{2.4}$$

with $\tilde{f}_t(x) = \sigma^2 t x^{1-\frac{2}{t}}$, $g_t(x) = 1 - \sigma^2 x^{-2/t}$ and $C_t(x) \to \frac{2}{\sigma}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2\sigma^2}\right)$ as $x \to \infty$.

□

Applying the result of Lemma 2.2 and Corollary 1.7 [28], the following result holds.

Proposition 2.1. *Under the conditions of Lemma 2.2, we have $F_t(x) \in D(\Lambda)$, where $D(\Lambda)$ is the domain of $\Lambda(x) = \exp\{-\exp(-x)\}$.*

Then, our aim is to select the suitable normalizing constants which ensure that the distribution of maximum tends to its extreme value limit. A combination of (1.3) and (2.4), we obtain that $d_n = b_n^t$. It follows from (2.3) that

$$c_n = \tilde{f}_t(d_n) = \sigma^2 t b_n^{t(1-\frac{2}{t})} = \sigma^2 t b_n^{t-2}. \tag{2.5}$$

The following work is to find the special normalizing constants c_n and d_n for the case of powered index $t = 2$. Similarly, it is necessary to establish the distributional tail representation of X^2 with $X \sim MD$.

Lemma 2.3. *Assume that $t = 2$. Let $F_2(x)$ stand for the cdf of X^2 with $X \sim MD$. Then for large x, we get*

$$1 - F_2(x) = C_2(x) \exp\left\{ -\int_1^x \frac{g_2(u)}{\tilde{f}_2(u)} du \right\}, \tag{2.6}$$

where

$$C_2(x) \to \frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{1}{2\sigma^2}\right) \text{ as } x \to \infty,$$
$$g_2(x) = 1 + \frac{\sigma^4}{x^2} \to 1 \text{ as } x \to \infty,$$

and

$$\tilde{f}_2(x) = 2\sigma^2\left(1 + \frac{\sigma^2}{x}\right) \text{ with } \tilde{f}_2'(x) \to 0 \text{ as } x \to \infty. \tag{2.7}$$

Proof. Similar to the case of $t \neq 2$, we get

$$\begin{aligned}
1 - F_2(x) &= 2\frac{\sigma^2 f(x^{\frac{1}{2}})}{x^{\frac{1}{2}}}\left[1 + \sigma^2 x^{-1} - \sigma^4 x^{-2} + 3\sigma^6 x^{-3} + O(x^{-4})\right] \\
&= 2\frac{\sigma^2 f(x^{\frac{1}{2}})}{x^{\frac{1}{2}}}(1 + \sigma^2 x^{-1})\left[1 - \sigma^4 x^{-2}(1 + \sigma^2 x^{-1})^{-1} + 3\sigma^6 x^{-3}(1 + \sigma^2 x^{-1})^{-1} + O(x^{-4})\right] \\
&= \frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\exp\left[-\frac{x}{2\sigma^2} + \frac{1}{2}\log x + \log\left(1 + \frac{\sigma^2}{x}\right)\right]\left[1 - \sigma^4 x^{-2} + 4\sigma^6 x^{-3} + O(x^{-4})\right] \\
&= \frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{1}{2\sigma^2}\right)\exp\left(-\int_1^x \frac{g_2(u)}{\tilde{f}_2(u)}du\right)\left[1 - \sigma^4 x^{-2} + 4\sigma^6 x^{-3} + O(x^{-4})\right]
\end{aligned} \tag{2.8}$$

with $g_2(x) = 1 + \sigma^4 t^{-2}$ and $\tilde{f}_2(x) = 2\sigma^2(1 + \sigma^2 t^{-1})$, where the third equality follows from the fact that $(1 + x)^a = 1 + ax + (a(a-1)/2)x^2 + O(x^3)$ for all $a \in \mathbb{R}$, as $x \to 0$.

□

Similar to the case of $t \neq 2$, we have the following result:

Proposition 2.2. *Under the assumptions of Lemma* 2.3*, we get* $F_2(x) \in D(\Lambda)$*, where* $D(\Lambda)$ *is the domain of* $\Lambda(x) = \exp\{-\exp(-x)\}$.

Now we discuss how to find the constants $c_n,\ d_n$. Analogous to the case of $t \neq 2$, we may make choice of $d_n = b_n^2$ and $c_n = \tilde{f}_2(d_n) = 2\sigma^2(1 + \sigma^2 b_n^{-2})$. Inspired by c_n, now change

$$\begin{aligned}
\bar{d}_n &= b_n^2 + 2\sigma^4 b_n^{-2}, \\
\bar{c}_n &= \tilde{f}_2(\bar{d}_n) \\
&= 2\sigma^2[1 + \sigma^2 b_n^{-2} - 2\sigma^6 b_n^{-6} + O(b_n^{-10})] \\
&\sim 2\sigma^2(1 + \sigma^2 b_n^{-2}).
\end{aligned} \tag{2.9}$$

Let

$$T_n(x,t) = F^{n-1}((c_n x + d_n)^{1/t}) - (1 - F((c_n x + d_n)^{1/t}))^{n-1}.$$

The following lemmas present the expansions of the two terms of densities of $(|M_n|^t - d_n)/c_n$.

Lemma 2.4. *For normalizing constants* c_n *and* d_n *determined by (2.5) and* $0 < t \neq 2$*, we have*

$$T_n(x,t) = \Lambda(x)\left\{1 - A_1(t,x)e^{-x}b_n^{-2} + \left(\frac{1}{2}A_1^2(t,x)e^{-x} - A_2(t,x)\right)e^{-x}b_n^{-4} + O(b_n^{-6})\right\} \tag{2.10}$$

as $n \to \infty$, where

$$A_1(t,x) = \sigma^2 \left(1 + x + \frac{(t-2)x^2}{2}\right) \tag{2.11}$$

and

$$A_2(t,x) = \sigma^4 \left(\frac{(t-2)^2 x^4}{8} + \frac{1}{6}(t-2)(5-2t)x^3 - \frac{x^2}{2} - x - 1\right). \tag{2.12}$$

Proof. Let $\delta_n(x,t) = (c_n x + d_n)^{1/t}$. One can easily see that $c_n x + d_n > 0$ for large n and fixed $x \in \mathbb{R}$. By (1.3), for large n, we have $b_n^2 \sim 2\sigma^2 \log n$. Then, by (2.5), we have

$$\delta_n^a(x,t) = b_n^a \left[1 + \frac{a\sigma^2 x}{b_n^2} + \frac{a(a-t)\sigma^4 x^2}{2b_n^4} + \frac{a(a-t)(a-2t)\sigma^6 x^3}{6b_n^6} + O(b_n^{-8})\right], \tag{2.13}$$

where it follows from the fact that

$$(1+x)^a = 1 + ax + (a(a-1)/2)x^2 + (a(a-1)(a-2)/6)x^3 + O(x^4),$$

for $a \in \mathbb{R}$, as $x \to 0$. Then, we get

$$\begin{aligned}
\frac{\sigma^2 f(\delta_n(x,t))}{\delta_n(x,t)} &\overset{\text{(a)}}{=} \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} b_n \left[1 + \frac{\sigma^2 x}{b_n^2} + \frac{(1-t)\sigma^4 x^2}{2b_n^4} + \frac{(1-t)(1-2t)\sigma^6 x^3}{6b_n^6} + O(b_n^{-8})\right] \\
&\quad \times \exp\left\{-\frac{b_n^2}{2\sigma^2}\left[1 + \frac{2\sigma^2 x}{b_n^2} + \frac{(2-t)\sigma^4 x^2}{b_n^4} + \frac{(2-t)(2-2t)\sigma^6 x^3}{3b_n^6} + O(b_n^{-8})\right]\right\} \\
&\overset{\text{(b)}}{=} \frac{\sigma^2 f(b_n)}{b_n} e^{-x} \left[1 + \frac{\sigma^2 x}{b_n^2} + \frac{(1-t)\sigma^4 x^2}{2b_n^4} + O(b_n^{-6})\right] \\
&\quad \times \left[1 - \frac{(2-t)\sigma^2 x^2}{2b_n^2} - \frac{(2-t)(1-t)\sigma^4 x^3}{3b_n^4} + \frac{(2-t)^2\sigma^4 x^4}{8b_n^4} + O(b_n^{-6})\right] \\
&\overset{\text{(c)}}{=} n^{-1} e^{-x} \left\{1 + \frac{\sigma^2 x}{b_n^2}\left(1 + \frac{1}{2}(t-2)x\right)\right. \\
&\quad \left. + \frac{\sigma^4 x^2}{b_n^4}\left[\frac{1}{8}(t-2)^2 x^2 + \frac{1}{6}(t-2)(5-2t)x + \frac{1-t}{2}\right] + O(b_n^{-6})\right\}
\end{aligned} \tag{2.14}$$

where (a) follows from (2.13) with $a = 1$ and 2, (b) is from the fact that $e^x = 1 + x + x^2/2 + O(x^3)$, as $x \to 0$ and (c) is due to (1.3). Furthermore, we get

$$\begin{aligned}
&1 + \sigma^2 \delta_n^{-2}(x,t) - \sigma^4 \delta_n^{-4}(x,t) + O(\delta_n^{-6}(x,t)) \\
&\overset{\text{(a)}}{=} 1 + \frac{\sigma^2}{b_n^2}\left[1 - \frac{2\sigma^2 x}{b_n^2} + O(b_n^{-4})\right] - \frac{\sigma^4}{b_n^4}\left[1 + O(b_n^{-2})\right] + O(b_n^{-6}) \\
&= 1 + \frac{\sigma^2}{b_n^2} - \frac{\sigma^4}{b_n^4}(1 + 2x) + O(b_n^{-6}),
\end{aligned} \tag{2.15}$$

where (a) is from (2.13) with $a = -2$ and -4. By Lemma 2.1, we get

$$\begin{aligned}
1 - F(\delta_n(x,t)) &= \frac{\sigma^2 f(\delta_n(x,t))}{\delta_n(x,t)} \left[1 + \sigma^2 \delta_n^{-2}(x,t) - \sigma^4 \delta_n^{-2}(x,t) + O(\delta_n^{-6}(x,t))\right] \\
&\overset{\text{(a)}}{=} n^{-1} e^{-x} \left\{1 + \frac{\sigma^2}{b_n^2}\left[1 + x + \frac{1}{2}(t-2)x^2\right]\right.
\end{aligned}$$

$$+\frac{\sigma^4}{b_n^4}\left[\frac{1}{8}(t-2)^2x^4+\frac{1}{6}(t-2)(5-2t)x^3-\frac{x^2}{2}-x-1\right]+O(b_n^{-6})\Bigg\}$$
$$=: n^{-1}e^{-x}\left[1+A_1(t,x)b_n^{-2}+A_2(t,x)b_n^{-4}+O(b_n^{-6})\right], \tag{2.16}$$

where (a) is due to (2.14) and (2.15). Accordingly,

$$\begin{aligned}F^{n-1}(\delta_n(x,t)) &= \exp\{(n-1)\log[1-(1-F(\delta_n(x,t)))]\}\\ &\overset{(a)}{=} \Lambda(x)\exp\left[-A_1(t,x)e^{-x}b_n^{-2}-A_2(t,x)e^{-x}b_n^{-4}+O(b_n^{-6})\right]\\ &\overset{(b)}{=} \Lambda(x)\left\{1-A_1(t,x)e^{-x}b_n^{-2}+\left(\frac{1}{2}A_1^2(t,x)e^{-x}-A_2(t,x)\right)e^{-x}b_n^{-4}+O(b_n^{-6})\right\},\end{aligned} \tag{2.17}$$

and

$$(1-F(\delta_n(x,t)))^{n-1}=\left\{\frac{e^{-x}}{n}\left[1+O(b_n^{-2})\right]\right\}^{n-1}=o(b_n^{-\eta}),\ \eta\geq 6, \tag{2.18}$$

where (a) is from the fact that $\log(1-x)=-x+O(x^2)$, as $x\to 0$, and (b) follows from that Taylor's expansion of e^x. The desired result follows by (2.17) and (2.18).

□

Lemma 2.5. *For the normalizing constants c_n and d_n determined by (2.5) and $0<t\neq 2$, we have*

$$\begin{aligned}n\frac{d}{dx}F((c_nx+d_n)^{1/t}) &= e^{-x}\Bigg\{1+\frac{\sigma^2x}{b_n^2}[3-t-(2-t)x]\\ &+\frac{\sigma^4x^2}{b_n^4}\left[\frac{1}{2}(3-t)(3-2t)+(t-2)\left(\frac{11}{6}-\frac{5}{6}t\right)x+\frac{1}{8}(t-2)^2x^2\right]+O(b_n^{-6})\Bigg\},\end{aligned} \tag{2.19}$$

as $n\to\infty$.

Proof. It is not hard to check that

$$n\frac{d}{dx}F((c_nx+d_n)^{1/t})=\frac{1}{t}nc_n(c_nx+d_n)^{1/t-1}f((c_nx+d_n)^{1/t}).$$

Therefore, we get

$$\begin{aligned}n\frac{d}{dx}F((c_nx+d_n)^{1/t}) &\overset{(a)}{=} \frac{n}{\sigma}b_n\sqrt{\frac{2}{\pi}}\left[1+\frac{(3-t)\sigma^2x}{b_n^2}+\frac{(3-t)(3-2t)\sigma^4x^2}{2b_n^4}+O(b_n^{-6})\right]\\ &\times\exp\left\{-\frac{b_n^2}{2\sigma^2}\left[1+\frac{2\sigma^2x}{b_n^2}+\frac{(2-t)\sigma^4x^2}{b_n^4}+\frac{(2-t)(2-2t)\sigma^6x^3}{3b_n^6}+O(b_n^{-8})\right]\right\}\\ &\overset{(b)}{=} nf(b_n)\frac{\sigma^2}{b_n}e^{-x}\left[1+\frac{(3-t)\sigma^2x}{b_n^2}+\frac{(3-t)(3-2t)\sigma^4x^2}{2b_n^4}+O(b_n^{-6})\right]\\ &\times\left[1-\frac{(2-t)\sigma^2x^2}{2b_n^2}-\frac{(2-t)(1-t)\sigma^4x^3}{3b_n^4}+\frac{(2-t)^2\sigma^4x^4}{8b_n^4}+O(b_n^{-6})\right]\\ &\overset{(c)}{=} e^{-x}\Bigg\{1+\frac{\sigma^2x}{b_n^2}[3-t-(2-t)x]\end{aligned}$$

$$+\frac{\sigma^4x^2}{b_n^4}\left[\frac{1}{2}(3-t)(3-2t)+(t-2)\left(\frac{11}{6}-\frac{5}{6}t\right)x+\frac{1}{8}(t-2)^2x^2\right]+O(b_n^{-6})\bigg\},$$

where (a) follows from (2.13) with $a=3-t$, (2.5) and (2.14) for the expansion of $f(\delta_n(x,t))$ with $\delta_n(x,t)=(c_nx+d_n)^{1/t}$, (b) is from the fact that $e^x=1+x+x^2/2+O(x^3)$, as $x\to 0$ and (c) is due to (1.3). The proof is complete.

□

Lemma 2.6. *For the normalizing constants c_n and d_n determined by (2.9) and $t=2$, we have*

$$T_n(x,t)=\Lambda(x)\left[1-B_1(t,x)e^{-x}b_n^{-4}-B_2(t,x)e^{-x}b_n^{-6}+O(b_n^{-8})\right], \tag{2.20}$$

as $n\to\infty$, where

$$B_1(t,x)=-\sigma^4\left(x^2+x+\frac{1}{2}\right) \tag{2.21}$$

and

$$B_2(t,x)=\sigma^6\left(\frac{4}{3}x^3+2x^2-2x+\frac{7}{3}\right). \tag{2.22}$$

Proof. The proof of the case of $t=2$ is similar to the case of $0<t\neq 2$. Note that $c_n=2\sigma^2(1+\sigma^2b_n^{-2})$, $d_n=b_n^2+2\sigma^4b_n^{-2}$ for $t=2$. So, we get

$$\delta_n(x,2)=(c_nx+d_n)^{1/2}=b_n[1+2\sigma^2b_n^{-2}x+2\sigma^4(x+1)b_n^{-4}]^{1/2}=:\beta_n.$$

Then, we have

$$\beta_n^a=b_n^a\left[1+\frac{a\sigma^2x}{b_n^2}+\frac{a\sigma^4}{b_n^4}\left(1+x-\frac{2-a}{2}x^2\right)-\frac{a(2-a)\sigma^6x}{b_n^6}\left(1+x-\frac{4-a}{6}x^2\right)+O(b_n^{-8})\right]. \tag{2.23}$$

Further, we get

$$\begin{aligned}
\frac{\sigma^2 f(\beta_n)}{\beta_n}&\overset{\text{(a)}}{=}\sqrt{\frac{2}{\pi}}\frac{b_n^2}{\sigma^3}\exp\left(-\frac{b_n^2}{2\sigma^2}\right)\frac{\sigma^2}{b_n}e^{-x}\\
&\quad\times\left[1+\frac{\sigma^2x}{b_n^2}+\frac{\sigma^4}{b_n^4}\left(1+x-\frac{1}{2}x^2\right)-\frac{\sigma^6x}{b_n^6}\left(1+x-\frac{1}{2}x^2\right)+O(b_n^{-8})\right]\\
&\quad\times\left[1-\frac{\sigma^2(1+x)}{b_n^2}+\frac{\sigma^4(1+x)^2}{2b_n^4}-\frac{\sigma^6(1+x)^3}{6b_n^6}+O(b_n^{-8})\right]\\
&\overset{\text{(b)}}{=}n^{-1}e^{-x}\left[1-\frac{\sigma^2}{b_n^2}-\frac{\sigma^4}{b_n^4}\left(x^2-x-\frac{3}{2}\right)+\frac{\sigma^6}{b_n^6}\left(\frac{4x^3}{3}-x^2-3x-\frac{7}{6}\right)+O(b_n^{-8})\right],
\end{aligned} \tag{2.24}$$

where (a) is from (2.23) with $a=1$ and 2 and $e^x=1+x+x^2/2+O(x^3)$, as $x\to 0$, and (b) is due to (1.3). Besides, applying (2.23) with $a=-2$, -4 and -6, we get

$$\begin{aligned}
&1+\sigma^2\beta_n^{-2}-\sigma^4\beta_n^{-4}+3\sigma^6\beta_n^{-6}+O(\beta_n^{-8})\\
=&1+\sigma^2b_n^{-2}\left[1-\frac{2\sigma^2x}{b_n^2}-\frac{2\sigma^4}{b_n^4}\left(1+x-2x^2\right)+O(b_n^{-6})\right]
\end{aligned}$$

$$-\sigma^4 b_n^{-4}\left[1-\frac{4\sigma^2 x}{b_n^2}+O(b_n^{-4})\right]+3\sigma^6 b_n^{-6}(1+O(b_n^{-2}))+O(b_n^{-8})$$
$$=1+\frac{\sigma^2}{b_n^2}-\frac{\sigma^4}{b_n^4}(2x+1)+\frac{\sigma^6}{b_n^6}(4x^2-2x+1)+O(b_n^{-8}). \tag{2.25}$$

Combining with Lemma 2.1, (2.24) and (2.25), we get

$$\begin{aligned}1-F(\beta_n)&=n^{-1}e^{-x}\left[1-\frac{\sigma^4}{b_n^4}\left(x^2+x+\frac{1}{2}\right)+\frac{\sigma^6}{b_n^4}\left(\frac{4}{3}x^3+2x^2-2x+\frac{7}{3}\right)+O(b_n^{-8})\right]\\&=: n^{-1}e^{-x}\left[1+B_1(t,x)b_n^{-4}+B_2(t,x)b_n^{-6}+O(b_n^{-8})\right].\end{aligned} \tag{2.26}$$

The remainder proof is the same as the case of $0<t\neq 2$. We omit it. The proof is complete.

□

Lemma 2.7. *For the normalizing constants c_n and d_n determined by (2.9) and $t=2$, we have*

$$n\frac{d}{dx}F((c_n x+d_n)^{1/t})=e^{-x}\left\{1-\frac{\sigma^4}{b_n^4}\left(x^2-x-\frac{1}{2}\right)+\frac{\sigma^6}{b_n^6}\left(\frac{4}{3}x^3-2x^2-2x+\frac{1}{3}\right)+O(b_n^{-8})\right\},$$

as $n\to\infty$.

Proof. By (2.24) and after observing that $c_n=2\sigma^2(1+\sigma^2 b_n^{-2})$, we get

$$\begin{aligned}n\frac{d}{dx}F(\beta_n)&=e^{-x}\left(1+\frac{\sigma^2}{b_n^2}\right)\left[1-\frac{\sigma^2}{b_n^2}-\frac{\sigma^4}{b_n^4}\left(x^2-x-\frac{3}{2}\right)+\frac{\sigma^6}{b_n^6}\left(\frac{4x^3}{3}-x^2-3x-\frac{7}{6}\right)+O(b_n^{-8})\right]\\&=e^{-x}\left\{1-\frac{\sigma^4}{b_n^4}\left(x^2-x-\frac{1}{2}\right)+\frac{\sigma^6}{b_n^6}\left(\frac{4}{3}x^3-2x^2-2x+\frac{1}{3}\right)+O(b_n^{-8})\right\}.\end{aligned} \tag{2.27}$$

The proof is complete.

□

As we mentioned in the introduction, Liu and Liu [21] obtained the pointwise convergence rate of distribution of partial maximum to its limiting distribution. Their main results are stated as follows.

Theorem 2.1. *Suppose that $\{X_n, n\geq 1\}$ is a sequence of i.i.d. random variables with cdf MD. Then,*

$$F^n(\hat{a}_n x+\hat{b}_n)-\Lambda(x)\sim\Lambda(x)e^{-x}\frac{(\log(2\log n))^2}{16\log n}, \tag{2.28}$$

for large n, where

$$\hat{a}_n=\frac{\sigma}{(2\log n)^{1/2}} \text{ and } \hat{b}_n=(2\sigma^2\log n)^{1/2}+\frac{\sigma\log(2\log n)+\sigma\log\frac{2}{\pi}}{2(2\log n)^{1/2}}. \tag{2.29}$$

3 Main result

In this section, we establish the higher-order expansions of the cdf and the pdf of powered maximum from MD sample.

Theorem 3.1.
(i) For $0 < t \neq 2$ and the normalizing constants c_n and d_n given by (2.5), we have

$$\mathbb{P}(|M_n|^t \leq c_n x + d_n) = \Lambda(x)\left\{1 - e^{-x}A_1(t,x)b_n^{-2} + e^{-x}\left[\frac{1}{2}e^{-x}A_1^2(t,x) - A_2(t,x)\right]b_n^{-4} + O(b_n^{-6})\right\}, \tag{3.1}$$

where

$$A_1(t,x) = \sigma^2\left[1 + x + \frac{1}{2}(t-2)x^2\right] \tag{3.2}$$

and

$$A_2(t,x) = \sigma^4\left[\frac{1}{8}(t-2)^2x^4 + \frac{1}{6}(t-2)(5-2t)x^3 - \frac{x^2}{2} - x - 1\right]. \tag{3.3}$$

(ii) For $t = 2$ and the normalizing constants c_n and d_n given by (2.9), we have

$$\mathbb{P}(|M_n|^t \leq c_n x + d_n) = \Lambda(x)\left[1 - e^{-x}B_1(t,x)b_n^{-4} - e^{-x}B_2(t,x)b_n^{-6} + O(b_n^{-8})\right], \tag{3.4}$$

where

$$B_1(t,x) = -\sigma^4\left(x^2 + x + \frac{1}{2}\right) \tag{3.5}$$

and

$$B_2(t,x) = \sigma^6\left(\frac{4}{3}x^3 + 2x^2 - 2x + \frac{7}{3}\right). \tag{3.6}$$

Remark 3.1. *From Theorem 3.1, one can easily see that the convergence rates of powered maximum of cdf for MD are proportional to $1/\log n$ and $1/(\log n)^2$ for power index $0 < t \neq 2$ and $t = 2$, respectively, since $1/b_n^2 \sim 2\sigma^2 \log n$ by (1.3).*

Remark 3.2. *From Theorems 2.1 and 3.1 (ii), we can observe that the convergence speed of powered extreme of cdf for MD is better than that of extreme of cdf.*

In the following we provide the higher-order expansions of the pdf of powered maximum.

Theorem 3.2.
(i) For $0 < t \neq 2$ and the normalizing constants c_n and d_n given by (2.5), we have

$$\frac{d}{dx}\mathbb{P}(|M_n|^t \leq c_n x + d_n) = \Lambda'(x)\left[1 + P_1(t,x)b_n^{-2} + P_2(t,x)b_n^{-4} + O(b_n^{-6})\right], \tag{3.7}$$

where

$$P_1(t,x) = \sigma^2\left\{-\left[\frac{(t-2)x^2}{2} + x + 1\right]e^{-x} + (t-2)x^2 - (t-3)x\right\}$$

and

$$P_2(t,x) = \sigma^4\left\{\frac{1}{2}\left[\frac{(t-2)x^2}{2} + x + 1\right]^2 e^{-2x}\right.$$

$$- \left[\frac{5(t-2)x^4}{8} - (t-2)\left(\frac{5}{6}t - \frac{10}{3}\right)x^3 + \left(2t + \frac{1}{2}\right)x^2 - 1\right]e^{-x}$$
$$+ \frac{(t-2)^2x^3}{8} - (t-2)\left(\frac{5}{6}t - \frac{11}{6}\right)x^2 + \frac{(t-3)(2t-3)}{2}x\Bigg\}.$$

(ii) For $t = 2$ and the normalizing constants c_n and d_n given by (2.9), we have

$$\frac{d}{dx}\mathbb{P}(|M_n|^t \le c_n x + d_n) = \Lambda'(x)\left[1 + Q_1(t,x)b_n^{-4} + Q_2(t,x)b_n^{-6} + O(b_n^{-8})\right], \tag{3.8}$$

where

$$Q_1(t,x) = \sigma^4\left[\left(x^2 + x + \frac{1}{2}\right)e^{-x} - x^2 + x + \frac{1}{2}\right]$$

and

$$Q_2(t,x) = -\sigma^6\left[\left(\frac{4}{3}x^3 + 2x^2 - 2x + \frac{7}{3}\right)e^{-x} - \frac{4}{3}x^3 + 2x^2 + 2x - \frac{1}{3}\right].$$

Remark 3.3. *From Theorem 3.2, it is not difficult to observe that the convergence speeds of powered extreme of pdf for MD are the same order of $1/\log n$ and $1/(\log n)^2$ for power index $0 < t \neq 2$ and $t = 2$, respectively, because of $1/b_n^2 \sim 2\sigma^2 \log n$ by (1.3).*

Remark 3.4. *For $t = 2$, the normalizing constants c_n and d_n are not given by (2.9), but we choose them as follows:*

$$c_n = 2\sigma^2(1 - \sigma^2 b_n^{-2}) \text{ and } d_n = b_n^2 - 2\sigma^4 b_n^{-2}, \tag{3.9}$$

then we derive

$$\begin{aligned}
&\mathbb{P}(|M_n|^t \le c_n x + d_n)\\
=&\Lambda(x)\Bigg\{1 - \frac{2e^{-x}\sigma^2}{b_n^2}(x+1) + \frac{e^{-x}\sigma^4}{b_n^4}\left[2e^{-x}(x+1)^2 - x^2 - x - \frac{3}{2}\right]b_n^{-4}\\
&- \frac{e^{-x}\sigma^6}{b_n^6}\left[\frac{4}{3}e^{-2x}(x+1)^3 - 2e^{-x}(x+1)\left(x^2 + x + \frac{3}{2}\right)\right.\\
&\left. + \frac{2}{3}x^3 + 2x^2 + 3x + \frac{14}{3}\right] + O(b_n^{-8})\Bigg\}
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&\frac{d}{dx}\mathbb{P}(|M_n|^t \le c_n x + d_n)\\
&= \Lambda'(x)\Bigg\{1 - \frac{2\sigma^2}{b_n^2}[e^{-x}(x+1) - x] + \frac{\sigma^4}{b_n^4}\left[2e^{-2x}(x+1)^2 - \left(5x^2 + 5x + \frac{3}{2}\right)e^{-x}\right.\\
&\left. + x^2 - x + \frac{1}{2}\right] + \frac{\sigma^6}{b_n^6}\left[4x(x+1)^2e^{-2x} - (4x^3 + 2x^2 + 2x + 1)e^{-x} + \frac{2}{3}x^3 - x - \frac{7}{6}\right] + O(b_n^{-8})\Bigg\}.
\end{aligned} \tag{3.11}$$

Obviously, the convergence rates of the cdf and the pdf of powered extreme given by (3.4) and (3.8), which are proportional to $1/(\log n)^2$, are faster than that given by (3.10) and (3.11). Consequently, the normalizing constants c_n and d_n determined by (2.9) are optimal.

4 Proof of main result

Proof of Theorem 3.1. By some fundamental calculations, we get

$$\mathbb{P}(|M_n|^t \le c_n x + d_n) = F^n((c_n x + d_n)^{1/t}) - (1 - F((c_n x + d_n)^{1/t}))^n. \tag{4.1}$$

First, we consider the case of $0 < t \neq 2$. By (2.16) and similar discussions as for (2.17) and (2.18), we get

$$F^n(\delta_n(x,t)) = \Lambda(x)\left\{1 - A_1(t,x)e^{-x}b_n^{-2} + \left(\frac{1}{2}A_1^2(t,x)e^{-x} - A_2(t,x)\right)e^{-x}b_n^{-4} + O(b_n^{-6})\right\}, \tag{4.2}$$

where $A_1(t,x)$ and $A_2(t,x)$ are determined by (2.11) and (2.12), and

$$(1 - F(\delta_n(x,t)))^n = \left\{\frac{e^{-x}}{n}\left[1 + O(b_n^{-2})\right]\right\}^n = o(b_n^{-\eta}),\ \eta \ge 6. \tag{4.3}$$

A combination of (4.2) and (4.3) implies that (3.1) holds.

For the case of $t = 2$, by similar arguments as for $0 < t \neq 2$, the desired result follows. The proof is complete.

□

Proof of Theorem 3.2. One can easily check that

$$\begin{aligned}\frac{d}{dx}\mathbb{P}(|M_n|^t \le c_n x + d_n) =& n\left(\frac{d}{dx}F((c_n x + d_n)^{1/t})\right) \\ &\times\left\{F^{n-1}((c_n x + d_n)^{1/t}) + [1 - F((c_n x + d_n)^{1/t})]^{n-1}\right\}.\end{aligned} \tag{4.4}$$

For $0 < t \neq 2$, combining with Lemmas 2.4 and 2.5, we get

$$\begin{aligned}\frac{1}{\Lambda'(x)}\frac{d}{dx}\mathbb{P}(|M_n|^t \le c_n x + d_n) - 1 =& \left\{1 + \frac{\sigma^2 x}{b_n^2}[3 - t - (2-t)x]\right.\\
&\left. + \frac{\sigma^4 x^2}{b_n^4}\left[\frac{1}{2}(3-t)(3-2t) + (t-2)\left(\frac{11}{6} - \frac{5}{6}t\right)x + \frac{1}{8}(t-2)^2x^2\right] + O(b_n^{-6})\right\}\\
&\times\left\{1 - \sigma^2\left[1 + x + \frac{1}{2}(t-2)x^2\right]e^{-x}b_n^{-2} + \left(\frac{1}{2}\left[1 + x + \frac{1}{2}(t-2)x^2\right]^2 e^{-x}\right.\right.\\
&\left.\left. - \left[\frac{1}{8}(t-2)^2x^4 + \frac{1}{6}(t-2)(5-2t)x^3 - \frac{x^2}{2} - x - 1\right]\right)\sigma^4 e^{-x}b_n^{-4} + O(b_n^{-6})\right\} - 1\\
=& \frac{\sigma^2}{b_n^2}\left\{-\left[\frac{(t-2)x^2}{2} + x + 1\right]e^{-x} + (t-2)x^2 - (t-3)x\right\}\\
&+ \frac{\sigma^4}{b_n^4}\left\{\frac{1}{2}\left[\frac{(t-2)x^2}{2} + x + 1\right]^2 e^{-2x}\right.\\
&- \left[\frac{5(t-2)x^4}{8} - (t-2)\left(\frac{5}{6}t - \frac{10}{3}\right)x^3 + \left(2t + \frac{1}{2}\right)x^2 - 1\right]e^{-x}\\
&\left. + \frac{(t-2)^2x^3}{8} - (t-2)\left(\frac{5}{6}t - \frac{11}{6}\right)x^2 + \frac{(t-3)(2t-3)}{2}x\right\} + O(b_n^{-6})\\
=& P_1(t,x)b_n^{-2} + P_2(t,x)b_n^{-4} + O(b_n^{-6}),\end{aligned}$$

which deduces (3.7).

The following is for the case of $t = 2$. By (4.4) and Lemmas 2.6 and 2.7, we gain

$$
\begin{aligned}
&\frac{1}{\Lambda'(x)}\frac{d}{dx}\mathbb{P}(|M_n|^t \le c_n x + d_n) - 1 = \left\{1 - \frac{\sigma^4}{b_n^4}\left(x^2 - x - \frac{1}{2}\right)\right.\\
&\quad\left. + \frac{\sigma^6}{b_n^6}\left(\frac{4}{3}x^3 - 2x^2 - 2x + \frac{1}{3}\right) + O(b_n^{-8})\right\}\\
&\quad \times\left[1 + \frac{\sigma^4 e^{-x}}{b_n^4}\left(x^2 + x + \frac{1}{2}\right) - \frac{\sigma^6 e^{-x}}{b_n^6}\left(\frac{4}{3}x^3 + 2x^2 - 2x + \frac{7}{3}\right) + O(b_n^{-8})\right] - 1\\
&=\frac{\sigma^4}{b_n^4}\left[\left(x^2 + x + \frac{1}{2}\right)e^{-x} - x^2 + x + \frac{1}{2}\right]\\
&\quad - \frac{\sigma^6}{b_n^6}\left[\left(\frac{4}{3}x^3 + 2x^2 - 2x + \frac{7}{3}\right)e^{-x} - \frac{4}{3}x^3 + 2x^2 + 2x - \frac{1}{3}\right] + O(b_n^{-8})\\
&=Q_1(t,x)b_n^{-4} + Q_2(t,x)b_n^{-6} + O(b_n^{-8}),
\end{aligned}
$$

which proves (3.8). The proof of Theorem 3.2 is finished.

□

Acknowledgments

Funding

This work was supported by Natural Science Foundation of China [grant number 61673015], [grant number 61273020] and Fundamental Research Funds for the Central Universities [grant number XDJK2015A007], Youth Science and technology talent development project (No.Qian jiao he KY zi [2018]313), Science and technology Foundation of Guizhou province [grant number Qian ke he Ji Chu [2016]1161], Guizhou province natural science foundation in China [grant number Qian Jiao He KY [2016]255].

References

[1] Smith, LR: Uniform rates of convergence in extreme-value theory. Adv. Appl. Probab. 14, 600-622 (1982)

[2] Leadbetter, MR, Lindgren, G, Rootzén, H: Extremes and Related Properties of Random Sequences and Processes. Springer, New York (1983)

[3] Galambos, J. (1987). *The asympotic theory of extreme order statistics.* (Second Edition) New York, Wiley.

[4] de Haan, L, Resnick, SI: Second-order regular variation and rates of convergence in extreme-value theory. Ann. Probab. 1, 97-124 (1996)

[5] Hall, P., 1979. On the rate of convergence of normal extremes. J. Appl. Probab. 16, 433-439.

[6] Hall, P., 1980. Estimating probabilities for normal extremes. Adv. Appl. Probab. 12, 491-500.

[7] Nair, K. A. (1981). Asymptotic distribution and moments of normal extremes. Annals of Probability, 9, 150-153.

[8] Liao, X., & Peng, Z. (2012). Convergence rates of limit distribution of maxima of lognormal samples. Journal of Mathematical Analysis and Applications, 395, 643-653.

[9] Lin, F., Zhang, X., Peng, Z., & Jiang, Y. (2011). On the rate of convergence of stsd extremes. Communications in Statistics - Theory and Methods, 40(10), 1795-1806.

[10] Lin, F., Peng, Z., & Yu, K. (2016). Convergence rate of extremes for the generalized short-tailed symmetric distribution. Bulletin of the Korean Mathematical Society, 53(5), 1549-1566.

[11] Du, L., & Chen, S. (2016). Asymptotic properties for distributions and densities of extremes from generalized gamma distribution. Journal of the Korean Statistical Society, 45(2), 188-198.

[12] Chen, S., & Du, L. (2017). Asymptotic expansions of density of normalized extremes from logarithmic general error distribution. Communications in Statistics, 46(7), 3459-3478.

[13] Huang, J., Wang, J., & Luo, G. (2017a). On the rate of convergence of maxima for the generalized maxwell distribution. Statistics A Journal of Theoretical & Applied Statistics, 1-13.

[14] Zhou, W., Ling, C. (2016). Higher-order expansions of powered extremes of normal samples. Statistics and Probability Letters, 111, 12-17.

[15] Liao, X, Peng, Z, Nadarajah, S: Asymptotic expansions of the moments of skew-normal extremes. Stat. Probab. Lett. 83, 1321-1329 (2013)

[16] Jia, P, Liao, X, Peng, Z: Asymptotic expansions of the moments of extremes from general error distribution. J. Math. Anal. Appl. 422, 1131-1145 (2015)

[17] Mandl, F. (2008). *Statistical Physics (2nd Edition)*. New Jersey: John Wiley & Sons.

[18] Shim, J. W., & Gatignol, R. (2013). How to obtain higher-order multivariate hermite expansion of maxwellCboltzmann distribution by using taylor expansion? Zeitschrift Fr Angewandte Mathematik Und Physik, 64(3), 473-482.

[19] Tomer, S.K. and Panwar, M.S. (2015). Estimation procedures for Maxwell distribution under type-I progressive hybrid censoring scheme. Journal of Statistical Computation and Simulation, 85(2), 339-356.

[20] Shim, J. W. (2017). Parametric lattice boltzmann method. Journal of Computational Physics, 338.

[21] Liu, C., Liu, B. (2013). Convergence rate of extremes from Maxwell sample. J. Inequal. Appl. 2013: 477. Available at: http://www.journalofinequalitiesandapplications.com/content/2013/1/477.

[22] Huang, J., Chen, S. Tail behavior of the generalized Maxwell distribution. Communications in Statistics-Theory and Methods. 2016, 45(14): 4230-4236.

[23] Dar, A.A., A. Ahmed and J.A. Reshi. Bayesian analysis of Maxwell-Boltzmann distribution under different loss functions and prior distributions. Pak. J. Statist. 2017, 33(6), 419-440

[24] Huang, J., Wang, J., Luo, G., & He, J. Tail properties and approximate distribution and expansion for extreme of LGMD. Journal of Inequalities & Applications, 2017b, 2017(1):1-16.

[25] Huang, J., Wang, J., Luo, G. Pu, H. Higher order expansion for moments of extreme for generalized Maxwell distribution, Communications in Statistics-Theory and Methods, 2018, 47(14): 3441-3452.

[26] Huang J.W., Wang J.J. On asymptotic of extremes from generalized Maxwell distribution. Bulletin of the Korean Mathematical Society, 2018a, 55(3): 679-698.

[27] Huang J.W., Wang J.J. Higher order asymptotic behaviour of partial maxima of random sample from generalized Maxwell distribution under power normalization. Applied Mathematics-A Journal of Chinese Universities, 2018b,33(2): 177-187.

[28] Resnick, S.I. (1987). *Extreme Value, Regular Variation, and Point Processes.* New York: Springer.