

Deng Entropy in Thermodynamics

Fan Liu^a, Yong Deng^{b,*}

^a*Yingcai Honors of School & School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, 610054, China*

^b*Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu, 610054, China*

Abstract

Dempster-Shafer theory (D-S theory) has been widely used in many fields. Recently, a new entropy called Deng entropy was proposed in D-S theory. As an extension of Shannon entropy, it can deal with uncertainty problems in D-S theory. Entropy originated in physics and was later widely used in many fields. A natural question is what is the form of Deng entropy in physics? In this paper, we proposed the Deng entropy in thermodynamics, and under the conditions of a given system, deduced the Deng entropy in thermodynamics. In addition, we discussed the properties of Deng entropy in thermodynamics. First, the Deng entropy of thermodynamics is an extension of Gibbs entropy, just as Deng entropy is an extension of Shannon's entropy. Similarly, Deng entropy in thermodynamics is also a measure of uncertainty. Given the state distribution of particles in a system, we can describe the uncertainty of particle states through Deng entropy in thermodynamics. Then, by proof, we find that Deng entropy in thermodynamics does not satisfy additivity. Finally, we also derived the probability distribution corresponding to the system when the Deng entropy in thermodynamics reaches its extreme value.

Keywords: D-S theory, Deng entropy, Uncertainty, Shannon entropy, Thermodynamics.

*Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu, 610054, China

Email address: dengentropy@uestc.edu.cn, prof.deng@hotmail.com (Yong Deng)

1. Introduction

Recently, a new entropy called Deng entropy was proposed in D-S theory. As an extension of Shannon entropy, it could handle uncertainty including non-specificity and discord which are consisting in a basic probability assignment (BPA). It has been widely used in many applications, such as, decision making [1, 2, 3], uncertainty [4, 5, 6], and data fusion [7, 8, 9]. For example, Wen *et al.* [10, 11, 12] use the dimension of information to identify important nodes in complex networks, and used the dimension of information to reduce the uncertainty of information. Entropy was firstly proposed in physics. Various types of entropy have been developed including information entropy [13, 14], Tsallis entropy [15], Rényi entropy [16], and so on [17, 18], which have many applications [19, 20, 21, 22]. Information entropy was derived from the Boltzmann-Gibbs (BG) entropy [23]. The maximum entropy is usually related to the uncertainty of the system [24]. Shannon entropy is a measure of uncertainty in information theory. Some mathematics in information theory were developed for thermodynamics [25, 26, 27]. There are connections between information entropy and thermodynamic entropy [28, 29, 30]. The measure defined in [31] is similar to Shannon entropy [13].

Deng entropy is a generalization of Shannon entropy, when BPA degenerates into probability, Deng entropy [32] degenerates into Shannon entropy [13]. Deng entropy is a tool can measure uncertainty of BPA in D-S theory. Uncertainty has been an open issue [33, 34]. For example, Gao *et al.* [35] explained the relationship between Pseudo-Pascal triangle and maximum Deng entropy. Pan *et al.* [36] used the information of entropy to study the probability conversion. Zhou *et al.* [37] used belief functions for medical diagnosis applications and risk evaluation [38]. It plays an important role in real world. Uncertainty usually comes from incomplete information, ignorance, *etc.* To deal with uncertainty, a lot of theories have been developed, such as, fuzzy sets [39], D-S theory [40, 41], quantum model [42], and entropy [13, 9, 43]. For example, as an extension of probability theory, D-S theory has flexibility in handling uncertain information. This flexibility comes from the expansion of the basic event space. Assuming that there are red balls and green balls in two boxes A_1 and A_2 , the number of balls in the two boxes is unknown, we randomly select a ball from the two boxes, the probability of getting the red ball is 0.6, and the probability of getting the green ball is 0.4. However, when we change this situation and replace some of the green balls in A_2 with

red balls, it is difficult for us to model the current situation with probability theory. We can use evidence theory to describe, at this time we can obtain a BPA: $m(R) = 0.6$, $m(R, G) = 0.4$, this kind of expression of uncertainty has become a reasonable way. In D-S theory, five types of uncertainty requirements are defined in [44]: probability consistency, set consistency, range, subadditive, additivity [44]. Not all the uncertain measurements satisfying these five requirements. Abellán [4] points out that Deng entropy [32] does not satisfy additivity and sub-additiveness and Tsallis entropy [15] does not satisfy additivity [45].

The concept of entropy is derived from physics, usually in thermodynamics, Gibbs entropy is usually involved. Shannon entropy was inspired by Gibbs entropy and was mentioned in the information paper. However, Deng entropy is an extension of Shannon entropy. A natural question was raised, what is the form of Deng entropy in thermodynamics, as Deng entropy has been widely used in various fields [46, 47]. If we can explore its meaning in physics, it can help us better understand Deng entropy and expand its applications. In this paper, we proposed the Deng entropy in thermodynamics and discussed the Deng entropy in thermodynamics. To the best of our knowledge, this is the first research involving the exploration of the physical meaning of Deng entropy. Deng entropy in thermodynamics also has the role of uncertainty measurement. We will discuss the properties of Deng entropy in thermodynamics later.

The contributions of this paper are as follows:

- First, we propose the Deng entropy in thermodynamics, and derive the Deng entropy in thermodynamics under the conditions of a given system.
- Then, we discussed the properties of Deng entropy in thermodynamics. Deng entropy in thermodynamics can be regarded as an extension of Gibbs entropy, and it is pointed out that Deng entropy in thermodynamics can measure the uncertainty of particles.
- We have discussed the additivity of Deng entropy in thermodynamics, and found that Deng entropy in thermodynamics does not satisfy additivity.
- Finally, we prove the probability distribution of particles when the Deng entropy in thermodynamics reaches the extreme value.

This paper is presented as follows. Some preliminaries are introduced in Section 2. In section 3, we proposed the Deng entropy in thermodynamics, and deduced the Deng entropy in thermodynamics under the conditions of a given system. Section 4 discusses the properties of Deng entropy in thermodynamics. and the final conclusion is given in Section 5.

2. Preliminaries

In this section, some preliminaries are briefly introduced including Deng entropy [32], Shannon entropy [13], and Gibbs entropy [23].

2.1. Deng entropy

D-S theory was developed by Dempster and Shafer [40, 41]. It offers a useful tool for uncertainty information. Deng entropy originated from D-S theory [48, 49, 50]. Hence, it is necessary to study uncertainty of D-S theory. Some basic knowledge in D-S theory is given as follows.

Definition 1. *Frame of discernment*

Let Ψ be a countable set of n elements called the frame of discernment which denotes a finite nonempty set of mutually exclusive and exhaustive hypotheses. Assume $\Psi = \{A_1, A_2, A_3, \dots, A_n\}$, the power set of Ψ is denoted with $P(\Psi)$ which contains all the possible subsets of it and composed of 2^n elements.

Definition 2. *Mass function*

A basic probability assignment (BPA) is a mass function. The range is from 0 to 1, which is defined by [40]

$$m : m(A) \mapsto [0, 1] \tag{1}$$

The mass $m(A)$ indicates the strength of the BPAs support for A , while $m(\Psi)$ is represented as the uncertainty of BPA. If $m(\phi) = 0$, any useful information can't be got from the BPA.

where $A \in P(\Psi)$ and it should satisfy the following conditions:

$$\sum_{A \in P(\Psi)} m(A) = 1, m(\phi) = 0 \tag{2}$$

Definition 3. *Deng entropy in D-S theory*

Deng entropy [32] is defined as follows.

$$E_d = \sum_i m(A_i) \log \frac{m(A_i)}{2^{|A_i|} - 1} \quad (3)$$

where A_i is a proposition of BPA, and $|A_i|$ is the cordiality of A_i .

Another form of Deng entropy is as follows.

$$E_d = \sum_i m(A_i) \log m(A_i) - \sum_i m(A_i) \log(2^{|A_i|} - 1) \quad (4)$$

According to discuss in [32], the term $\sum_i m(A_i) \log m(A_i)$ represents the disorder of BPA among focal elements, and the term $-\sum_i m(A_i) \log(2^{|A_i|} - 1)$ is a measure of total non-specificity in the BPA.

2.2. Gibbs entropy

In thermodynamics, Gibbs entropy is usually involved. Given macro constraints, Gibbs entropy can be obtained from the probabilities of different microscopic states of the system, and its form is as follows [23].

Definition 4.

$$S_G = -k \sum_{s=1}^W \rho_s \ln \rho_s \quad (5)$$

Where k is Boltzmann's constant, W is the number of all microscopic states in the system, and ρ_s is the probability of the corresponding microscopic state s .

2.3. Shannon entropy

In information theory, Shannon entropy is usually used as a measure of information uncertainty in information systems based on entropy in physics, the definition of Shannon entropy is as follows [13].

Definition 5.

$$E_S = - \sum_{s=1}^N \rho_s \ln \rho_s \quad (6)$$

Where N is the number of all basic events and ρ_s is the probability of the basic event s .

3. Deng Entropy in Thermodynamics

Deng entropy can be used as a measure of the uncertainty of the belief function in evidence theory. This measure of uncertainty is established in the frame of discernment. The frame of discernment in evidence theory is an extension of the basic events in probability theory. The Gibbs entropy is derived from the probabilistic modeling of the different microscopic states of the system. Inspired by this idea, can the different microscopic states of the system be expanded to derive the Deng entropy in thermodynamics?

In this section, we will introduce Deng entropy in thermodynamics. Assuming that the original set of micro-states in the system is $S = \{s_1, s_2, \dots, s_W\}$, now the micro-states are extended to the new set $\Omega = P(S) \setminus \{\phi\}$, where $P(S) = \{\{\phi\}, \{s_1\}, \dots, \{s_W\}, \{s_1, s_2\}, \dots, \{s_1, s_2, \dots, s_i\}, \dots, \{S\}\}$ is the power set of S .

In order to derive Deng entropy in thermodynamics, the measurement function $f(Y)$ is defined as follows.

$$f(Y) = |Y| = i, \quad Y = \{s_1, s_2, \dots, s_i\} \in \Omega \quad (7)$$

Then the definition of Deng entropy in thermodynamics is as follows.

$$S_D = -k \sum_{s, s \in \Omega} \rho_s \ln \frac{\rho_s}{2^{|s|} - 1} \quad (8)$$

Next, we will try to derive Deng entropy from a system under given conditions. Given a closed system, there are N particles in the system, and the average number of particles in a quantum state s with energy ε_s is as follows.

$$f_s = (2^{|s|} - 1) \exp(-\alpha - \beta\varepsilon_s) \quad (9)$$

Therefore, the probability ρ_s that the particle is in the quantum state s is

$$\rho_s = \frac{f_s}{N} = \frac{(2^{|s|} - 1) \exp(-\alpha - \beta\varepsilon_s)}{N} \quad (10)$$

According to Appendix, we have $\frac{1}{Z} = \frac{\exp(-\alpha)}{N}$, the variant form of Eq. (10) six is as follows.

$$-\ln\left(\frac{\rho_s}{2^{|s|} - 1}\right) = \ln Z + \beta\varepsilon_s \quad (11)$$

Obviously, the probability ρ_s that the particle is in the quantum state s satisfies the normalization condition.

$$\sum_{s,s \in \Omega} \rho_s = 1 \quad (12)$$

The average energy E of particles in the system can be expressed as

$$E = \sum_{s,s \in \Omega} \rho_s \varepsilon_s \quad (13)$$

According to Appendix, the entropy of the system can be expressed as

$$\begin{aligned} S &= Nk(\ln Z - \beta \frac{\partial}{\partial \beta} \ln Z) \\ &= Nk(\ln Z + \beta \bar{\varepsilon}) \end{aligned} \quad (14)$$

Then the Deng entropy of the system in thermodynamics can be expressed as follows.

$$\begin{aligned} S_D &= Nk(\ln Z - \beta \frac{\partial}{\partial \beta} \ln Z) \\ &= Nk(\ln Z + \beta E) \\ &= Nk(\ln Z + \beta \sum_{s,s \in \Omega} \rho_s \varepsilon_s) \\ &= Nk(\sum_{s,s \in \Omega} \rho_s (\ln Z + \beta \varepsilon_s)) \\ &= -Nk(\sum_{s,s \in \Omega} \rho_s \ln(\frac{\rho_s}{2^{|s|} - 1})) \end{aligned} \quad (15)$$

Table 1 shows the comparison of Deng entropy [32], Gibbs entropy [23], Deng entropy in thermodynamics, and Shannon entropy [13]. First of all, it can be seen that Gibbs entropy and Shannon entropy are formally consistent in mathematical form. Deng entropy and Deng entropy in thermodynamics are also the same in mathematical form. Secondly, Deng entropy can be regarded as an extension of Shannon entropy. Later we will prove that Deng entropy in thermodynamics is also an extension of Gibbs entropy. Finally, we can see that Shannon entropy is defined in the frame of discernment,

Table 1: Comparison of Deng entropy in thermodynamics , Gibbs entropy, Deng entropy and Shannon entropy

Name	Formula	Space
Gibbs entropy [23]	$S_G = -k \sum_{s=1}^W \rho_s \ln \rho_s$	S
Shannon entropy [13]	$E_S = - \sum_{s=1}^N \rho_s \ln \rho_s$	Ψ
Deng entropy in thermodynamics	$S_D = -k \sum_{s,s \in \Omega} \rho_s \ln \frac{\rho_s}{2^{ s -1}}$	Ω
Deng entropy [32]	$E_d = - \sum_i m(A_i) \log \frac{m(A_i)}{2^{ A_i -1}}$	$P(\Psi)$

and Deng entropy is defined in the power set of the frame of discernment. However, Gibbs entropy is defined on the set of all microscopic states of the system, and Deng entropy in thermodynamics is defined on the extended set of microscopic states.

4. The Properties of Deng Entropy in Thermodynamics

In this section, we will discuss the properties of Deng entropy in thermodynamics and analyze its potential applications.

Proposition 1. *When the microscopic state Ω of the system degenerates into the microscopic state S , the Deng entropy in thermodynamics degenerates into Gibbs entropy.*

Proof 1. *Obviously, when the system satisfies the state S , there is*

$$\begin{aligned}
 S_D &= -k \sum_{s,s \in S} \rho_s \ln \frac{\rho_s}{2^{|s|-1}} \\
 &= -k \sum_{s,s \in S} \rho_s \ln \rho_s \\
 &= S_G
 \end{aligned} \tag{16}$$

Example 1. *Eq. (15) shows that the Deng entropy in thermodynamics of a particle in the system is $S_D = -k \sum_{s,s \in \Omega} \rho_s \ln \frac{\rho_s}{2^{|s|-1}}$, which depends on the probability ρ that the particle is in each possible state Ω . It may be assumed that the particle is in a specific state s_1 at this time, and $|s_1| = 1$, then the Deng entropy in thermodynamics of the particle at this time is zero and the Gibbs entropy of the particle is also zero. In contrast. If the particle may be in a different state, the Deng entropy of the particle is not zero, which shows*

that Deng entropy in thermodynamics can be used as a measure of uncertain information. When we know less about the known information, the greater the Deng entropy in thermodynamics.

Abellán analyzed in [4] that Deng entropy is not additivity, so does Deng entropy in thermodynamics satisfy additivity? Next we will prove that the Deng entropy in thermodynamics does not satisfy additivity.

Proposition 2. *Suppose a given system $(M + N)$ is composed of two subsystems M and N (the interaction between the systems is ignored). The probability distribution ρ_{M+N} of the system $M + N$ is equal to the product $\rho_M \cdot \rho_N$ of the probabilities of the two subsystems M and N . If there is the entropy S_{M+N} of the system $(M+N)$ is equal to the sum of the entropy $S_M + S_N$ of the two subsystems M and N , then we call the entropy S to be additive. It can be proved that Deng entropy in thermodynamics is not additive, that is $S_{M+N} \neq S_M + S_N$.*

Proof 2. *Obviously we have*

$$\sum_{s_M} \rho_{s_M} = 1 \quad (17)$$

$$\sum_{s_N} \rho_{s_N} = 1 \quad (18)$$

Therefore,

$$\sum_{s_M} \sum_{s_N} \rho_{s_{M+N}} = \sum_{s_M} \sum_{s_N} \rho_{s_M} \cdot \rho_{s_N} = \sum_{s_M} \rho_{s_M} \cdot \sum_{s_N} \rho_{s_N} = 1 \quad (19)$$

The thermodynamic Deng entropy of the system $(M + N)$ is

$$\begin{aligned} S_{M+N} &= -k \sum_{s_M} \sum_{s_N} \rho_{M+N} \ln \frac{\rho_{s_{M+N}}}{2^{|s_{M+N}|} - 1} \\ &= -k \sum_{s_M} \sum_{s_N} \rho_M \cdot \rho_N \ln \frac{\rho_M \cdot \rho_N}{2^{|s_{M+N}|} - 1} \\ &\neq -k \sum_{s_M} \sum_{s_N} \rho_M \cdot \rho_N \ln \frac{\rho_M \cdot \rho_N}{(2^{|s_M|} - 1) \cdot (2^{|s_N|} - 1)} \\ &= -k \sum_{s_M} \rho_M \ln \frac{\rho_M}{(2^{|s_M|} - 1)} - k \sum_{s_N} \rho_N \ln \frac{\rho_N}{(2^{|s_N|} - 1)} \\ &= S_M + S_N \end{aligned} \quad (20)$$

Proposition 3. *If the number of particles N and the volume V of the system, and the energy E are constant, the probability distribution when the Deng entropy in the thermodynamics of the system reaches the extremum value is*

$$\rho_s = (2^{|s|} - 1) \exp\left(\frac{\gamma}{k} - (2^{|s|} - 1)\right) \quad (21)$$

Where γ is a constant and k is Boltzmann's constant.

Proof 3. *Assuming that N_s and E_s respectively represent the number and energy of the system when the system is in state s , the average value of the number and energy is*

$$\bar{N} = \sum_s N_s \rho_s \quad (22)$$

$$\bar{E} = \sum_s E_s \rho_s \quad (23)$$

Lets suppose that ρ_s has a change in δ , then in Eq. (8), the change in S_D is

$$\delta S_D = -k \sum_s \left(\ln \frac{\rho_s}{2^{|s|} - 1} + 2^{|s|} - 1 \right) \delta \rho_s \quad (24)$$

There are conditions.

$$\begin{aligned} N_s &= N \\ E_s &= E \\ \sum_s \delta \rho_s &= 0 \end{aligned} \quad (25)$$

Introduce the constant γ , so there is

$$\sum_s \left[-k \left(\ln \frac{\rho_s}{2^{|s|} - 1} + 2^{|s|} - 1 \right) + \gamma \right] \delta \rho_s = 0 \quad (26)$$

So we have

$$\rho_s = (2^{|s|} - 1) \exp\left(\frac{\gamma}{k} - (2^{|s|} - 1)\right) \quad (27)$$

5. Conclusion

Deng entropy is an extension of Shannon entropy that was proposed in D-S theory. The concept of entropy originated from physics. In thermodynamics, Gibbs entropy is often mentioned. It is worth noting that Shannon entropy in information theory was inspired by Gibbs entropy. Recently, Deng entropy has been widely used in the measurement of uncertainty of information. With the gradual exploration, researches on Deng entropy's divergence and maximum Deng entropy have been involved. Thinking further, what is the significance of Deng entropy in physics? These studies can help us better understand Deng Entropy and apply it in a wider range of fields.

To the best of our knowledge, the research of this paper is the first time to explore the physical meaning of Deng entropy. First of all, the Deng entropy in thermodynamics has been proposed. Given certain conditions of the system, the Deng entropy in thermodynamics has been derived. Then, we also discussed the properties of Deng entropy in thermodynamics and its potential applications. The Deng entropy in thermodynamics can be regarded as the expansion of Gibbs entropy, and the uncertainty of particles can be measured. The additivity of entropy is an important study for entropy. It is worth noting that not all entropy satisfies additivity. We discussed the additivity of Deng entropy in thermodynamics, and proved that Deng entropy in thermodynamics does not satisfy additivity. Finally, we discuss the distribution of the microscopic state of the system when the Deng entropy in thermodynamics reaches its extreme value.

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Appendix

This section shows the derivation process of entropy in thermodynamics. Some basic physical symbols are shown in Table 2.

Table 2: Basic physical symbols

Symbol	Physical meaning
Ω	The number of microstates.
ε	The energy of a system with discrete energy levels.
ω	The number of different states corresponding to a particular energy level.
a	The number of partivles at the energy level ε .
U	The total energy in a system.
Z	Partition functions in physics can describe the statistical of a system.
W	Work done on a system by an external force.
S	Entropy is an extensive property in a system.

Given a system, the definition of entropy is as follows.

$$S = \frac{dQ}{T} \quad (28)$$

Where dQ is the change of heat in the system and T is the temperature of the system.

In the Boltzmann system [51], the number of microscopic states of a particle can be expressed as :

$$\Omega_{M.B} = \frac{N!}{\prod_l a_l!} \prod_l (\omega_l)^{a_l} \quad (29)$$

Where $\Omega_{M.B}$ is the total number of microscopic states of the particle, a_l is the number of particles at the energy level ε_l , and ω_l is the degeneracy at the energy level ε_l .

The logarithm of both sides has

$$\ln \Omega = \ln N! - \sum_l \ln a_l! + \sum_l a_l \ln(\omega_l) \quad (30)$$

Eq. 30 can be simplified to

$$\begin{aligned} \ln \Omega &= N(\ln N - 1) - \sum_l a_l(\ln a_l - 1) + \sum_l a_l \ln(\omega_l) \\ &= N \ln N - \sum_l a_l \ln a_l + \sum_l a_l \ln(\omega_l) \end{aligned} \quad (31)$$

Suppose that a_l has δa_l changes. Then $\ln \Omega$ has the following changes

$$\delta \ln \Omega = - \sum_l \ln\left(\frac{a_l}{\omega_l}\right) \delta a_l \quad (32)$$

In order to make $\ln \Omega$ have large distribution, let $\delta \ln \Omega$ be equal to 0, then there is

$$\delta \ln \Omega = - \sum_l \ln\left(\frac{a_l}{\omega_l}\right) \delta a_l = 0 \quad (33)$$

But these δa_l are not independent, they must meet the following conditions

$$\delta N = \sum_l \delta a_l = 0 \quad \delta E = \sum_l \varepsilon_l \delta a_l = 0 \quad (34)$$

In the case where the constraint Eq. 34 is satisfied, Eq. 32 is equivalent to the following Eq. 35 regardless of the values of the parameters α and β .

$$- \sum_l \left(\ln\left(\frac{a_l}{\omega_l}\right) + \alpha + \beta \varepsilon_l \right) \delta a_l = 0 \quad (35)$$

Under the conditions given by Eq. 34, it makes the two δa_l cannot take any value. It is possible to set the two to determine δa_1 and δa_2 . Then α and β can be determined by the following equation.

$$\ln \frac{a_1}{\omega_1} + \alpha + \beta \varepsilon_1 \quad \ln \frac{a_2}{\omega_2} + \alpha + \beta \varepsilon_2 \quad (36)$$

Simply Eq. 35 to

$$\sum_{l=3} (\ln(\frac{a_l}{\omega_l}) + \alpha + \beta\varepsilon_l)\delta a_l = 0 \quad (37)$$

Since each δa_l in Eq. 37 can take values independently, the coefficient of δa_l in Eq. 37 is required to be equal to 0, that is :

$$\sum_{l=3} (\ln(\frac{a_l}{\omega_l}) + \alpha + \beta\varepsilon_l) = 0 \quad l = 1, 2, 3, \dots W \quad (38)$$

Eq. 39 can be derived from Eq. 38 and 36.

$$a_l = (\omega_l)e^{-\alpha-\beta\varepsilon_l} \quad (39)$$

In summary, the total number of particles is:

$$N = \sum_l (\omega_l)e^{-\alpha-\beta\varepsilon_l} \quad (40)$$

Internal energy is the statistical average of the total energy of the irregular motion of particles in the system. So internal energy can be expressed as

$$U = \sum_l a_l \varepsilon_l = \sum_l \varepsilon_l (\omega_l) e^{-\alpha-\beta\varepsilon_l} \quad (41)$$

Introduce a partition function whose expression is as follows:

$$Z = \sum_l (\omega_l) e^{-\beta\varepsilon_l} \quad (42)$$

Eq. 43 can be derived from Eq. 40 and 42.

$$N = e^{-\alpha} Z \quad (43)$$

Substituting Eq. 43 into Eq. 41, then eliminating α , Eq. 44 can be obtained as follows:

$$U = -N \frac{\partial}{\partial \beta} \ln Z \quad (44)$$

The change of the system's internal energy dU in the process is equal to the sum of the work dW done by the outside world on the system and the heat dQ absorbed by the system from the outside.

$$dU = dQ + dW \quad (45)$$

The force applied to a particle at the entropy of the energy level ε_l is $\frac{\partial \varepsilon_l}{\partial y}$. Therefore, the generalized force exerted by the outside world on the system is:

$$\begin{aligned}
Y &= \sum_l \frac{\partial \varepsilon_l}{\partial y} a_l \\
&= \sum_l \frac{\partial \varepsilon_l}{\partial y} (\omega_l) e^{-\alpha - \beta \varepsilon_l} \\
&= e^{-\alpha} \left(-\frac{1}{\beta} \frac{\partial}{\partial y} \right) \sum_l \frac{\partial \varepsilon_l}{\partial y} (\omega_l) e^{-\beta \varepsilon_l} \\
&= \frac{N}{Z} \left(-\frac{1}{\beta} \frac{\partial}{\partial y} \right) Z \\
&= -\frac{N}{\beta} \frac{\partial}{\partial y} \ln Z
\end{aligned} \tag{46}$$

Where Y is the generalized force of the external system corresponding to the external parameter y .

The heat absorbed by the system from the outside during the process is related to the process, so dQ is not a full differential but an infinitesimal amount. It is proved by the second law of thermodynamics that dQ has integral factor $\frac{1}{T}$, and multiplying $\frac{1}{T}$ by dQ gives the complete differential dS :

$$\frac{1}{T} dQ = \frac{1}{T} (dU - Y dy) = dS \tag{47}$$

Eq. 48 can be derived from Eq. 44, Eq. 44 and Eq. 47.

$$\begin{aligned}
dQ &= dU - Y dy \\
&= -N d \left(\frac{\partial \ln Z}{\partial \beta} \right) + \frac{N}{\beta} \frac{\partial \ln Z}{\partial y} dy
\end{aligned} \tag{48}$$

Eq. 48 is simultaneously multiplied by β :

$$\beta (dU - Y dy) = -N \beta \left(\frac{\partial \ln Z}{\partial \beta} \right) + N \frac{\partial \ln Z}{\partial y} dy \tag{49}$$

Because the partition function Z is a function of β and y , the full differential of $\ln Z$:

$$d \ln Z = \frac{\partial \ln Z}{\partial \beta} d\beta + \frac{\partial \ln Z}{\partial y} dy \tag{50}$$

Eq. 51 can be derived from Eq. 49 and Eq. 50.

$$\beta(dU - Ydy) = Nd(\ln Z - \beta \frac{\partial}{\partial \beta} \ln Z) \quad (51)$$

Then the expression of S can be derived from Eq. 28 and 51. Let $\beta = \frac{1}{kT}$

$$S = Nk(\ln Z - \beta \frac{\partial}{\partial \beta} \ln Z) \quad (52)$$