A Factorial Identity Resulting from the Orthogonality Relation of the Associated Laguerre Polynomials

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Abstract

Plugging the closed-form expression of the associated Laguerre polynomials into their orthogonality relation, the latter reduces to a factorial identity that takes a simple, non-trivial form for even-degree polynomials.

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1. Introduction

The associated Laguerre polynomials $L_\lambda^\nu(x)$ are λ-degree polynomial solutions to the associated Laguerre differential equation

$$xy''(x)+(\nu+1-x)y'(x)+\lambda y(x)=0$$

for $\nu,\lambda=0,1,...$ [1, 2]. If $\nu=0$, the associated Laguerre polynomials reduce to the Laguerre polynomials $L_\lambda(x)$ [1, 2].

The polynomials $L_\lambda^\nu(x)$ are given by the closed-form expression [1, 3]

$$L_\lambda^\nu(x)=\sum_{n=0}^{\lambda}\frac{(-1)^n}{n!}\binom{\lambda+\nu}{\lambda-n}x^n\ ,$$

where $\binom{\lambda+\nu}{\lambda-n}$ is the binomial coefficient, i.e.

$$\binom{\lambda+\nu}{\lambda-n}=\frac{(\lambda+\nu)!}{(\lambda-n)!(\nu+n)!}$$

The polynomials $L_\lambda^\nu(x)$ satisfy the orthogonality relation [1-3]

$$\int_0^\infty dxx^\nu\exp(-x)L_\lambda^\nu(x)L_{\lambda'}^\nu(x)=0\,,$$

for $\lambda\neq\lambda'$.

2. The factorial identity

Using the closed-form expression of the associated Laguerre polynomials, the orthogonality integral takes the form

$$\int_0^\infty dxx^\nu\exp(-x)L_\lambda^\nu(x)L_{\lambda'}^\nu(x)=$$

$$=\int_0^\infty dxx^\nu\exp(-x)\left(\sum_{m=0}^{\lambda}\frac{(-1)^m}{m!}\binom{\lambda+\nu}{\lambda-m}x^m\right)\left(\sum_{n=0}^{\lambda'}\frac{(-1)^n}{n!}\binom{\lambda'+\nu}{\lambda'-n}x^n\right)=$$

$$= \int_0^\infty dx x^\nu \exp(-x) \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}}{m!n!} \binom{\lambda+\nu}{\lambda-m} \binom{\lambda'+\nu}{\lambda'-n} x^{m+n} =$$

$$= \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}}{m!n!} \binom{\lambda+\nu}{\lambda-m} \binom{\lambda'+\nu}{\lambda'-n} \int_0^\infty dx x^{m+n+\nu} \exp(-x)$$

That is

$$\int_0^\infty dx x^\nu \exp(-x) L_\lambda^\nu(x) L_{\lambda'}^\nu(x) = \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}}{m!n!} \binom{\lambda+\nu}{\lambda-m} \binom{\lambda'+\nu}{\lambda'-n} \int_0^\infty dx x^{m+n+\nu} \exp(-x)$$

The integral on the right-hand side is easily calculated using the Gamma function, since

$$\int_0^\infty dx x^{m+n+\nu} \exp(-x) = \Gamma(m+n+\nu+1) = (m+n+\nu)!,$$

and then

$$\int_0^\infty dx x^\nu \exp(-x) L_\lambda^\nu(x) L_{\lambda'}^\nu(x) = \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}}{m!n!} \binom{\lambda+\nu}{\lambda-m} \binom{\lambda'+\nu}{\lambda'-n} (m+n+\nu)! =$$

$$= \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}}{m!n!} \frac{(\lambda+\nu)!}{(\lambda-m)!(m+\nu)!} \frac{(\lambda'+\nu)!}{(\lambda'-n)!(n+\nu)!} (m+n+\nu)! =$$

$$= (\lambda+\nu)!(\lambda'+\nu)! \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}(m+n+\nu)!}{m!n!(\lambda-m)!(\lambda'-n)!(m+\nu)!(n+\nu)!}$$

That is

$$\int_0^\infty dx x^\nu \exp(-x) L_\lambda^\nu(x) L_{\lambda'}^\nu(x) = (\lambda+\nu)!(\lambda'+\nu)! \sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}(m+n+\nu)!}{m!n!(\lambda-m)!(\lambda'-n)!(m+\nu)!(n+\nu)!}$$

Then, since $(\lambda+\nu)!(\lambda'+\nu)! \neq 0$, the orthogonality relation of the associated Laguerre polynomials reduces to the following factorial identity

$$\sum_{(m,n)=(0,0)}^{(\lambda,\lambda')} \frac{(-1)^{m+n}(m+n+\nu)!}{m!n!(\lambda-m)!(\lambda'-n)!(m+\nu)!(n+\nu)!} = 0 \tag{1}$$

where $\lambda, \lambda', \nu = 0,1,\ldots$ and $\lambda \neq \lambda'$.

If $\lambda' = 0$, then $n = 0$ too, and (1) reads

$$\sum_{m=0}^{\lambda\neq 0}\frac{(-1)^m(m+\nu)!}{m!0!(\lambda-m)!(0-0)!(m+\nu)!(0+\nu)!}=0\overset{0!=1}{\Rightarrow}\frac{1}{\nu!}\sum_{m=0}^{\lambda\neq 0}\frac{(-1)^m}{m!(\lambda-m)!}=0,$$

and since $1/\nu!$ is non-zero,

$$\sum_{m=0}^{\lambda\neq 0}\frac{(-1)^m}{m!(\lambda-m)!}=0 \qquad (2)$$

The series in (2) has $\lambda+1$ terms. If λ is odd, the series has an even number of terms, while m and $\lambda-m$ have different parity, i.e. if m is even/odd then $\lambda-m$ is odd/even, and thus

$$\frac{(-1)^{\lambda-m}}{(\lambda-m)!(\lambda-(\lambda-m))!}=\frac{(-1)^{\lambda-m}}{(\lambda-m)!m!}=-\frac{(-1)^m}{m!(\lambda-m)!}$$

Then, the terms with $m=0$ and $m=\lambda$ are opposite, as are the terms with $m=1$ and $m=\lambda-1$, as are the terms with $m=2$ and $m=\lambda-2$, etc. Thus, in this case, the series consists of $(\lambda+1)/2$ pairs of opposite terms, and the identity (2) is rather trivial. However, if λ is even, m and $\lambda-m$ have the same parity, and also the series has an odd number of terms, thus it does not consist of pairs of opposite terms. Therefore, in the case where λ is even, the identity (2) is not trivial. Moreover, setting $\lambda\to 2\lambda$, with $\lambda=1,2,...$, the series in (2) is written as

$$\sum_{m=0}^{2\lambda}\frac{(-1)^m}{m!(2\lambda-m)!}=\sum_{m=0}^{\lambda}\frac{(-1)^{2m}}{(2m)!(2\lambda-2m)!}+\sum_{m=1}^{\lambda}\frac{(-1)^{2m-1}}{(2m-1)!(2\lambda-(2m-1))!}=$$
$$=\sum_{m=0}^{\lambda}\frac{1}{(2m)!(2(\lambda-m))!}-\sum_{m=1}^{\lambda}\frac{1}{(2m-1)!(2(\lambda-m)+1)!},$$

and (2) takes the form

$$\sum_{m=0}^{\lambda}\frac{1}{(2m)!(2(\lambda-m))!}=\sum_{m=1}^{\lambda}\frac{1}{(2m-1)!(2(\lambda-m)+1)!} \qquad (3)$$

with $\lambda=1,2,...$

Let us verify (3) for $\lambda=1,2,3$.

For $\lambda=1$, we have

$$\sum_{m=0}^{1}\frac{1}{(2m)!\big(2(1-m)\big)!}=\frac{1}{0!2!}+\frac{1}{2!0!}=\frac{1}{2}+\frac{1}{2}=1$$

and

$$\sum_{m=1}^{1}\frac{1}{(2m-1)!\big(2(1-m)+1\big)!}=\frac{1}{1!\big(2(1-1)+1\big)!}=\frac{1}{1!1!}=1$$

For $\lambda=2$, we have

$$\sum_{m=0}^{2}\frac{1}{(2m)!\big(2(2-m)\big)!}=\frac{1}{0!4!}+\frac{1}{2!2!}+\frac{1}{4!0!}=\frac{2}{4!}+\frac{1}{2!2!}=\frac{1}{12}+\frac{1}{4}=\frac{1}{12}+\frac{3}{12}=\frac{4}{12}=\frac{1}{3}$$

and

$$\sum_{m=1}^{2}\frac{1}{(2m-1)!\big(2(2-m)+1\big)!}=\frac{1}{1!3!}+\frac{1}{3!1!}=\frac{2}{3!}=\frac{1}{3}$$

For $\lambda=3$, we have

$$\sum_{m=0}^{3}\frac{1}{(2m)!\big(2(3-m)\big)!}=\frac{1}{0!6!}+\frac{1}{2!4!}+\frac{1}{4!2!}+\frac{1}{6!0!}=\frac{2}{6!}+\frac{2}{2!4!}=\frac{1}{3*4*5*6}+\frac{1}{2*3*4}=$$
$$=\frac{1}{3*4*5*6}+\frac{15}{3*4*5*6}=\frac{16}{3*4*5*6}=\frac{4}{3*5*6}=\frac{2}{3*5*3}=\frac{2}{45}$$

and

$$\sum_{m=1}^{3}\frac{1}{(2m-1)!\big(2(3-m)+1\big)!}=\frac{1}{1!5!}+\frac{1}{3!3!}+\frac{1}{5!1!}=\frac{2}{5!}+\frac{1}{3!3!}=\frac{1}{3*4*5}+\frac{1}{2*3*2*3}=$$
$$=\frac{1}{3*4*5}+\frac{1}{3*4*3}=\frac{3}{3*4*5*3}+\frac{5}{3*4*3*5}=\frac{8}{3*4*5*3}=\frac{2}{3*5*3}=\frac{2}{45}$$

3. References

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[3] http://mathworld.wolfram.com/AssociatedLaguerrePolynomial.html.