Arithmetic of Analysis II
Absolute (Modulus) Value Function
Alternate Method of Solution

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In an ordered field, there is an important mapping $| - |$ called absolute value function. We shall define $| - |$ for $\mathbb{R}$.

The absolute value function is defined by $| - |: \mathbb{R} \to [0, \infty)$.

Remarks: The image $| \ a \ | \ \ of \ \ a \in \mathbb{R}$ is called the absolute of a. We have the following simple but important properties of the absolute value function.

1. $| \ a \ | \geq \ 0, \quad \forall \ \ a \in \mathbb{R}$
2. $| \ a \ | = \ 0, \quad iff \ \ a = 0$
3. $-a \ \ \leq | \ a \ | \quad and \quad a \ \ \leq | \ a \ |, \quad \forall \ \ a \in \mathbb{R}$
4. $- | \ a \ | \ \ \leq \ \ a \ \ \leq | \ a \ |, \quad \forall \ \ a \in \mathbb{R}$
5. $| \ ab \ | = | \ a \ || \ b \ |, \quad \forall \ \ a, b \in \mathbb{R}$
6. $| \ \frac{a}{b} \ | = \ \frac{|a|}{|b|}, \quad \forall \ \ a, b \in \mathbb{R}$
7. $a^2 = | \ a \ |^2, \quad \forall \ \ a \in \mathbb{R}$

Triangle Inequality

Let $a, b \in \mathbb{R}$. Show that $| \ a + b \ | \leq | \ a \ | + | \ b \ |$.
Proof:
$(| \ a + b \ |)^2 = (a + b)^2$

$= a^2 + 2ab + b^2 \quad = | \ a \ |^2 + 2ab + | \ b \ |^2 \quad \leq | \ a \ |^2 + 2 \ | \ ab \ | + | \ b \ |^2 \quad = | \ a \ |^2 + 2 \ | \ a \ || \ b \ | + | \ b \ |^2 \quad = (| \ a \ | + | \ b \ |)^2$

Take positive square root

$$| \ a + b \ | \leq | \ a \ | + | \ b \ |$$

Remarks:

1. Extension of the above theorem to complex number.
   Show that $| \ a + b \ | \leq | \ a \ | + | \ b \ |, \quad where \ \ a, b \in \mathbb{C}$.
   Proof:
   $(| \ a + b \ |)^2 \ \ \leq | \ a \ |^2 + 2 \ | \ ab \ | + | \ b \ |^2$

   $= | \ a \ |^2 + 2 \ | \ a \ || \ b \ | + | \ b \ |^2 \qquad = (| \ a \ | + | \ b \ |)^2$

   Take positive square root

   $$| \ a + b \ | \leq | \ a \ | + | \ b \ |$$

   Note: Some steps are missing because $\mathbb{C}$ is not an ordered field.

2. Extension of the above theorem to vector.
   Show that $| \ a + b \ | \leq | \ a \ | + | \ b \ |$, where a, b are vectors.
   Proof:
   $(| \ a + b \ |)^2 \ \ \leq | \ a \ |^2 + 2 \ | \ a \ || \ b \ | + | \ b \ |^2$

   $= (| \ a \ | + | \ b \ |)^2$

   Take positive square root

$$| a+b | \leq | a | + | b |$$

Note: Some steps are missing because vector is defined for dot and cross (undergraduate syllabus). In addition, vector is not ordered.

Further Examples

1. Show that $| a | - | b | \leq | a-b |$
   Proof:
   $(| a | - | b |)^2 =| a |^2 -2 | a || b | + | b |^2$

   $$=| a |^2 -2 | ab | + | b |^2 \quad = a^2 - 2 | ab | + b^2 \quad \leq a^2 - 2ab + b^2 \quad = (a-b)^2 \quad = (| a-b |)^2$$

   Take positive square root

   $$| a | - | b | \leq | a-b |$$

2. Show that $| | a | - | b | | \ \leq \ | a-b |$
   Proof:
   $(|| a | - | b ||)^2 = (| a | - | b |)^2$

   $$=| a |^2 -2 | a || b | + | b |^2 \quad =| a |^2 -2 | ab | + | b |^2 \quad \leq a^2 - 2ab + b^2 \quad = (a-b)^2 \quad = (| a-b |)^2$$

   Take positive square root

   $$| | a | - | b | | \ \leq \ | a-b |$$

Reference:

Robert Bartle and Donald Sherbert (2000). Introduction to Real Analysis. John Wiley and Sons, Inc.
Stephen Abbott (2000). Understanding Analysis. Springer.