Real Numbers in the Neighborhood of Infinity

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June 27, 2019

Abstract

We demonstrate the existence of a broad class of real numbers which are not elements of any number field: those in the neighborhood of infinity. After considering the reals and the affinely extended reals, we prove that numbers in the neighborhood of infinity are ordinary real numbers. As an application in complex analysis, we show that the Riemann zeta function has infinitely many non-trivial zeros off the critical line for every application in quantum field theory.

§1 Real Numbers

Definition 1.1 A real number $x \in \mathbb{R}$ is a cut in the real number line.

Definition 1.2 The real numbers are defined in interval notation as

$$\mathbb{R} \quad \equiv \quad (-\infty, \infty) \quad ,$$

where the linear interval $(-\infty, \infty)$ is an infinite line.

Definition 1.3 A cut in a line $x \in \mathbb{R}$ separates one line into two pieces as

$$\mathbb{R} \setminus x = (-\infty, x) \cup (x, \infty) \quad .$$

Remark 1.4 A number is a cut in a line. A line is defined a priori. All lines can be cut so all lines are number lines. A given line is the real line by definition. A real number separates the real number line into a set of "larger" real numbers and a set of "smaller" real numbers.

Definition 1.5 Call real numbers in the neighborhood of the origin $x \in \mathbb{R}_0$. Define them such that

$$x \in \mathbb{R}_0 \qquad \Longleftrightarrow \qquad x \in \mathbb{R} \quad , \quad \text{and} \quad -n < x < n \quad ,$$

for some $n \in \mathbb{N}$.

Definition 1.6 Call real numbers in the neighborhood of infinity $x \in \mathbb{R}_\infty$. Define them as all real numbers except for real numbers in the neighborhood of the origin:

$$\mathbb{R}_\infty \quad \equiv \quad \mathbb{R} \setminus \mathbb{R}_0 \quad .$$

Remark 1.7 The main result of this paper demonstrates that $\mathbb{R}_\infty$ is not an empty set.

Definition 1.8 For $x \in \mathbb{R}$ and $n, k \in \mathbb{N}$, we have the property

$$\lim_{x\to 0^\pm} \frac{1}{x} = \text{diverges} \quad , \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{k=1}^{n} k = \text{diverges} \quad .$$

§2 Affinely Extended Real Numbers

Definition 2.1 Define two affinely extended real numbers $\pm\infty$ such that for $x \in \mathbb{R}$ and $n, k \in \mathbb{N}$

$$\lim_{x\to 0^\pm} \frac{1}{x} = \pm\infty \quad , \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{k=1}^{n} k = \infty \quad .$$

Definition 2.2 The set of all affinely extended real numbers is

$$\overline{\mathbb{R}} \quad \equiv \quad \mathbb{R} \cup \{\pm\infty\} \quad .$$

Definition 2.3 The affinely extended real numbers are defined in interval notation as

$$\overline{\mathbb{R}} \quad \equiv \quad [-\infty, \infty] \quad .$$

Definition 2.4 An affinely extended real number $x \in \overline{\mathbb{R}}$ is $\pm\infty$ or it is a cut in the affinely extended real number line.

Definition 2.5 In $\overline{\mathbb{R}}$, $\pm\infty$ are such that the limit of any monotonic sequence of real numbers which diverges in $\mathbb{R}$ is equal to ∞ or $-\infty$.

Theorem 2.6 *If $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$ then $x \in \mathbb{R}$.*

Proof. Proof follows from Definition 2.2.

Definition 2.7 Infinity is such that

$$\infty - \infty = \text{undefined} \quad , \qquad \text{and} \qquad \frac{\infty}{\infty} = \text{undefined} \quad .$$

Definition 2.8 Infinity does not have the distributive property of multiplication. For two non-zero real numbers a and b we have

$$\infty \cdot \big(b + a\big) = \infty \quad .$$

To the contrary, we write for two orthogonal unit vectors $\hat{e}_1$ and $\hat{e}_2$

$$\infty \cdot \big(\hat{e}_1 + \hat{e}_2\big) = \text{undefined} \quad .$$

Remark 2.9 In Definition 2.8 the former case gives the appearance of a distributive property because we can sum $b + a$ and then use the multiplicative absorptive property of ∞ to obtain a simplified result but we cannot do so in the latter case which is only mentioned in anticipation of the expression $\infty \cdot \big(x + iy\big)$ which appears in Section 5.

§3 Modified Infinity

Definition 3.1 Additive absorption is a property of $\pm\infty$ such that non-zero numbers are additive identities of $\pm\infty$. The additive absorptive property is

$$\pm\infty \pm x = \pm\infty \mp x = \pm\infty \quad , \qquad \text{for} \qquad x \in \mathbb{R}_0 \quad , \quad x \neq 0 \quad .$$

Definition 3.2 Let the symbol $\widehat{\infty}$ be called modified infinity and endow it with every property of ∞ except additive absorption.

Definition 3.3 Infinity and modified infinity both describe the same affinely extended real number

$$\pm\widehat{\infty} = \pm\infty \qquad \Longrightarrow \qquad \overline{\mathbb{R}} \quad \equiv \quad [-\widehat{\infty}, \widehat{\infty}] \quad .$$

Definition 3.4 The hat which differentiates modified infinity $\pm\widehat{\infty}$ from canonical infinity $\pm\infty$ is inserted and removed by choice except in the case where it invokes a contradiction and must be removed by definition. If a contradiction is obtained via non-absorptivity then the hat must be removed to alleviate the contradiction.

Remark 3.5 When the $\pm\infty$ symbols appear as $\pm\widehat{\infty}$, consider the hat to be an instruction to delay the additive absorption of $\pm\infty$ indefinitely or until such delay causes a contradiction. The instruction to "delay additive absorption" should be understood to mean that additive absorption is not a property of $\pm\widehat{\infty}$ but that the additive absorptive property can be trivially implemented after an ad hoc decision to remove the hat by choice or after its removal is required by definition.

Example 3.6 An example of a statement in which the hat does not invoke a contradiction and may be left in place is

$$x = \widehat{\infty} - b \quad .$$

Example 3.7 An example of a statement in which the hat invokes a contradiction and may not be left in place is given by two sequences

$$x_n = \sum_{k=1}^{n} k \quad , \qquad \text{and} \qquad y_n = c_0 + \sum_{k=1}^{n} k \quad ,$$

where $n \in \mathbb{N}$ and c_0 is some non-zero real number. Since ∞ and $\widehat{\infty}$ are the same number we can use Definitions 2.1 and 3.2 to write

$$\lim_{n\to\infty} x_n = \infty = \widehat{\infty} \quad , \qquad \text{and} \qquad \lim_{n\to\infty} y_n = \infty = \widehat{\infty} \quad .$$

We may also write, however,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} c_0 + \lim_{n\to\infty} x_n = c_0 + \widehat{\infty} \quad .$$

This delivers an equality

$$\widehat{\infty} = c_0 + \widehat{\infty} \quad ,$$

which contradicts the delayed additive absorption of $\widehat{\infty}$. At this point, we must cease to delay additive absorption by removing the hat. Then

$$\infty = c_0 + \infty \quad ,$$

demonstrates the usual additive absorptive property of infinity and there is no contradiction.

Definition 3.8 $\widehat{\infty}$ is such that for any non-zero $b \in \mathbb{R}_0$

$$\begin{aligned}
\pm\widehat{\infty} + b &= b \pm \widehat{\infty} \\
\pm\widehat{\infty} - b &= -b \pm \widehat{\infty} \\
\pm\widehat{\infty} + \big(- b\big) &= \pm\widehat{\infty} - b \\
\pm\widehat{\infty} + b &= \pm\widehat{\infty} - \big(- b\big) \\
-\big(\pm \widehat{\infty}\big) &= \mp\widehat{\infty} \\
\pm\widehat{\infty} \cdot b = b \cdot \pm\widehat{\infty} &= \begin{cases} \pm\widehat{\infty} & \text{if} \quad b > 0 \\ \mp\widehat{\infty} & \text{if} \quad b < 0 \end{cases} \\
\frac{\pm\widehat{\infty}}{b} &= \begin{cases} \pm\widehat{\infty} & \text{if} \quad b > 0 \\ \mp\widehat{\infty} & \text{if} \quad b < 0 \end{cases} \\
\frac{b}{\pm\widehat{\infty}} &= 0 \quad .
\end{aligned}$$

Definition 3.9 $\widehat{\infty}$ is such that

$$\pm\widehat{\infty} + 0 = 0 \pm \widehat{\infty} = \pm\widehat{\infty} - 0 = \text{undefined}$$

$$\begin{aligned} \pm\widehat{\infty}\cdot 0 = 0\cdot\pm\widehat{\infty} &= \text{undefined} \\ \frac{\pm\widehat{\infty}}{0} &= \text{undefined} \\ \frac{0}{\pm\,\widehat{\infty}} &= 0 \quad . \end{aligned}$$

Remark 3.10 We will revisit the lack of an additive identity in Example 4.13.

Definition 3.11 $\pm\infty$ has all the properties assigned to $\pm\widehat{\infty}$ in Definitions 3.8 and 3.9 plus the additive absorptive operation of Definition 3.1.

§4 Real Numbers in the Neighborhood of Infinity

Definition 4.1 Let $\widehat{\mathbb{R}}$ be the set of all numbers of the form

$$x = \pm\big(\widehat{\infty} - b\big) \quad , \qquad \text{where} \qquad b \in \mathbb{R}_0 \quad , \quad b > 0 \quad .$$

Definition 4.2 The ordering of $\widehat{\mathbb{R}}$ numbers is such that

$$\begin{aligned} \pm\big(\widehat{\infty} - b\big) = \pm\big(\widehat{\infty} - a\big) &\qquad \Longleftrightarrow \qquad a = b \\ \big(\widehat{\infty} - b\big) > \big(\widehat{\infty} - a\big) &\qquad \Longleftrightarrow \qquad a > b \\ -\big(\widehat{\infty} - b\big) > -\big(\widehat{\infty} - a\big) &\qquad \Longleftrightarrow \qquad a < b \\ \big(\widehat{\infty} - b\big) > -\big(\widehat{\infty} - a\big) &\qquad \forall \qquad b, a \in \mathbb{R}_0 \\ \big(\widehat{\infty} - b\big) > x &\qquad \forall \qquad b, x \in \mathbb{R}_0 \\ -\big(\widehat{\infty} - b\big) < x &\qquad \forall \qquad b, x \in \mathbb{R}_0 \\ \big(\widehat{\infty} - b\big) < \infty &\qquad \forall \qquad b \in \mathbb{R}_0 \\ -\big(\widehat{\infty} - b\big) > -\infty &\qquad \forall \qquad b \in \mathbb{R}_0 \quad . \end{aligned}$$

Theorem 4.3 *All numbers $x \in \widehat{\mathbb{R}}$ are cuts in the affinely extended real number line, i.e.: they are affinely extended real numbers.*

Proof. By Definition 2.4, an affinely extended real number is a cut in or end-point of the affinely extended real number line. Definition 1.3 requires that a cut separates one line into two pieces. Observe that

$$\begin{aligned} \overline{\mathbb{R}} \setminus (\widehat{\infty} - b) &\quad \equiv \quad [-\infty, \widehat{\infty} - b) \cup (\widehat{\infty} - b, \infty] \\ \overline{\mathbb{R}} \setminus \big(-\widehat{\infty} + b\big) &\quad \equiv \quad [-\infty, -\widehat{\infty} + b) \cup (-\widehat{\infty} + b, \infty] \quad . \end{aligned}$$

All numbers $x \in \widehat{\mathbb{R}}$ conform to the definition of affinely extended real numbers.

Main Theorem 4.4 *All numbers $x \in \widehat{\mathbb{R}}$ are real numbers.*

Proof. If a number is an affinely extended real number $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$ then by Theorem 2.6 we have $x \in \mathbb{R}$. In the absence of additive absorption

$$\pm\big(\widehat{\infty} - b\big) \neq \pm\widehat{\infty} = \pm\infty \quad ,$$

because it is the definition of $\widehat{\mathbb{R}}$ that $b \neq 0$. Also note that

$$\mathbb{R}/(\widehat{\infty} - b) \quad \equiv \quad (-\infty, \widehat{\infty} - b) \cup (\widehat{\infty} - b, \infty) \quad .$$

All numbers $x \in \widehat{\mathbb{R}}$ satisfy Definition 1.1.

Theorem 4.5 *All numbers $x \in \widehat{\mathbb{R}}$ are real numbers in the neighborhood of infinity $x \in \mathbb{R}_\infty$.*

Proof. We have shown in Main Theorem 4.4 that

$$\widehat{\mathbb{R}} \subset \mathbb{R} \quad ,$$

so we will satisfy Definition 1.6 if we show that

$$\widehat{\mathbb{R}} \cap \mathbb{R}_0 \quad \equiv \quad \emptyset \quad .$$

Definition 1.5 requires that elements of $\mathbb{R}_0$ satisfy

$$-n < x < n \quad ,$$

so assume

$$-n < \pm\big(\widehat{\infty} - b\big) < n \quad .$$

Since $b \in \mathbb{R}_0$ we know it has an additive inverse. Add or subtract b to obtain

$$-n + b < \widehat{\infty} < n + b \quad , \qquad \text{and} \qquad -n - b < -\widehat{\infty} < n - b \quad .$$

We obtain a contradiction as ∞ cannot be less than the sum of two finite numbers and $-\infty$ cannot be greater than the difference of two finite numbers. All $\widehat{\mathbb{R}}$ numbers satisfy the definition of $\mathbb{R}_\infty$.

Remark 4.6 The remaining definitions in the section define the arithmetic operations for $\widehat{\mathbb{R}}$ numbers. The purpose in defining these operations is to supplement the canonical operations for $\mathbb{R}_0$ and $\infty \sim \widehat{\infty}$. Every $\widehat{\mathbb{R}}$ number can be decomposed and its pieces manipulated separately but the main purpose of defining special operations for $\widehat{\mathbb{R}}$ is to define new operations for expressions which are undefined under the arithmetic operations of $\mathbb{R}_0$ and ∞ alone or whose structure vanishes under additive absorption.

Definition 4.7 The arithmetic operations of $\widehat{\mathbb{R}}$ numbers with $\mathbb{R}_0$ numbers are

$$\begin{aligned}
-\big(\widehat{\infty}-b\big) &= -\widehat{\infty}+b \\
-\big(-\widehat{\infty}+b\big) &= \widehat{\infty}-b \\
\pm\big(\widehat{\infty}-b\big)+x = x\pm\big(\widehat{\infty}-b\big) &= \begin{cases} \pm\widehat{\infty}\mp\big(b-x\big) & \text{if} \quad b\neq x \\ \pm\widehat{\infty} & \text{if} \quad b=x \end{cases} \\
\pm\big(\widehat{\infty}-b\big)\cdot x = x\cdot\pm\big(\widehat{\infty}-b\big) &= \begin{cases} \pm\big(\widehat{\infty}-xb\big) & \text{if} \quad x\neq 0 \\ \text{undefined} & \text{if} \quad x=0 \end{cases} \\
\frac{\pm\big(\widehat{\infty}-b\big)}{x} &= \begin{cases} \pm\widehat{\infty}\mp\dfrac{b}{x} & \text{if} \quad x\neq 0 \\ \text{undefined} & \text{if} \quad x=0 \end{cases} \\
\frac{x}{\pm\big(\widehat{\infty}-b\big)} &= 0 \quad .
\end{aligned}$$

Theorem 4.8 *The quotient of a number $x\in\mathbb{R}_0$ divided by a number $y\in\widehat{\mathbb{R}}$ is identically zero.*

Proof. Let z be any non-zero real number such that

$$\frac{x}{y}=z \quad .$$

Since $||x||<||y||$, we have $||z||<1$ which implies $z\in\mathbb{R}_0$. All $\mathbb{R}_0$ numbers have a multiplicative inverse. We find, therefore, that

$$\frac{x}{zy}=1 \qquad \Longleftrightarrow \qquad x=zy \quad .$$

The hat on $\widehat{\infty}$ only suppresses additive absorption so

$$zy=z\cdot\pm\big(\widehat{\infty}-b\big)=\pm\big(\widehat{\infty}-zb\big) \quad .$$

This delivers a contradiction because it requires that x is a real number in the neighborhood of infinity while we have already defined it to be a real number in the neighborhood of the origin. Therefore, the only possible numerical value for x/y is 0.

Definition 4.9 The arithmetic operations of $\widehat{\mathbb{R}}$ numbers with $\widehat{\mathbb{R}}$ numbers are

$$\begin{aligned}
\pm\big(\widehat{\infty}-b\big)\pm\big(\widehat{\infty}-a\big) &= \pm\widehat{\infty}\mp\big(b+a\big) \\
\pm\big(\widehat{\infty}-b\big)\mp\big(\widehat{\infty}-a\big) &= \pm\big(a-b\big) \\
\big(\widehat{\infty}-b\big)\big(\widehat{\infty}-a\big) &= \text{undefined}
\end{aligned}$$

$$\frac{\widehat{\infty}-b}{\widehat{\infty}-a} = \text{undefined} \quad .$$

Theorem 4.10 *Products of the form $\widehat{\mathbb{R}}\cdot\widehat{\mathbb{R}}$ are undefined.*

Proof. Assume $\widehat{\infty}$ has the distributive property of multiplication so that

$$\big(\widehat{\infty}-b\big)\big(\widehat{\infty}-a\big) = \widehat{\infty}-\widehat{\infty}-\widehat{\infty}+ba \quad .$$

By different groupings of the terms, we obtain contradictory values:

$$\widehat{\infty}-\widehat{\infty}-\widehat{\infty}+ba = \widehat{\infty}-\Big[\widehat{\infty}+\big(\widehat{\infty}-ba\big)\Big] = \widehat{\infty}-\big(\widehat{\infty}-ba\big) = ba \quad ,$$

and

$$-\widehat{\infty}+\widehat{\infty}-\widehat{\infty}+ba = -\widehat{\infty}+\Big[\widehat{\infty}-\big(\widehat{\infty}-ba\big)\Big] = -\widehat{\infty}+ba \quad .$$

While these two values do not agree, the latter has the added problem of being negative number. If we assumed

$$\big(\widehat{\infty}-b\big)\big(\widehat{\infty}-a\big) = ba \quad , \qquad \text{then} \qquad \frac{1}{\big(\widehat{\infty}-b\big)\big(\widehat{\infty}-a\big)} = \frac{1}{ab} \quad ,$$

in contradiction to

$$\frac{1}{\big(\widehat{\infty}-b\big)\big(\widehat{\infty}-a\big)} = \left(\frac{1}{\widehat{\infty}-b}\right)\left(\frac{1}{\widehat{\infty}-a}\right) = 0 \quad .$$

Theorem 4.11 *Quotients of the form $\widehat{\mathbb{R}}/\widehat{\mathbb{R}}$ are undefined.*

Proof. Observe that

$$\frac{\widehat{\infty}-b}{\widehat{\infty}-a} = \frac{\widehat{\infty}}{\widehat{\infty}-a} - \frac{b}{\widehat{\infty}-a} \quad .$$

Insert the multiplicative identity into the first term so that

$$\frac{\widehat{\infty}\cdot 1}{\widehat{\infty}-a} = \widehat{\infty}\cdot\left(\frac{1}{\widehat{\infty}-a}\right) = \widehat{\infty}\cdot 0 = \text{undefined} \quad .$$

Remark 4.12 Although

$$\big(\widehat{\infty}-b\big)-\big(\widehat{\infty}-a\big) = a-b \quad ,$$

implies the existence of an additive inverse for every $\widehat{\mathbb{R}}$ number, this does not imply an additive inverse for ∞ because the case of $a=b=0$ is ruled out by the definition of $\widehat{\mathbb{R}}$.

Example 4.13 Definition 3.9 states that infinity does not have an additive identity element. An example motivating this condition is given by the limit

$$\lim_{x\to\infty}\big(x^2-x\big)=\infty$$
$$\lim_{x\to\infty}\big(x^2-x\big)=\lim_{x\to\infty}x^2-\lim_{x\to\infty}x=\infty-\infty \quad .$$

which is usually used to demonstrate the lack of an additive inverse for ∞. If infinity is bestowed with an additive inverse then we obtain a contradiction $\infty=0$. The expression $\infty-\infty$ is thus undefined. If we added the hats to infinity then we could insert the additive identity to write

$$\infty=\widehat{\infty}-\widehat{\infty}=\widehat{\infty}-\widehat{\infty}+0=\widehat{\infty}-\widehat{\infty}+1-1=\big(\widehat{\infty}-1\big)-\big(\widehat{\infty}-1\big)=0 \quad .$$

We see that unhatted infinity likewise cannot have zero as an additive identity because we could write

$$\infty=\infty-\infty=\infty-\infty+\big(1-1\big)=\widehat{\infty}-\widehat{\infty}+1-1=\big(\widehat{\infty}-1\big)-\big(\widehat{\infty}-1\big)=0 \quad .$$

where we have simply chosen not to do the additive absorptive operation within the freedom afforded to the order of algebraic operations. This example confirms that the only difference between ∞ and $\widehat{\infty}$ is an instruction to delay additive absorption for the latter.

Remark 4.14 The expressions ∞ and $\widehat{\infty}$ are perfectly well defined but $\infty+0$ and $\widehat{\infty}+0$ are examples of an undefined composition. Since ∞ is not an $\widehat{\mathbb{R}}$ number, this property cannot create problems for the algebra of $\widehat{\mathbb{R}}$ numbers. Essentially, we have traded the zero additive identity element of infinity for the freedom to add and subtract $\widehat{\mathbb{R}}$ numbers.

Theorem 4.15 *An $\widehat{\mathbb{R}}$ number does not have a multiplicative inverse.*

Proof. Assume

$$x\big(\widehat{\infty}-b\big)=1 \quad .$$

If $x\in\widehat{\mathbb{R}}$ then the operation is undefined. If $x\in\mathbb{R}_0$ then we obtain

$$\widehat{\infty}-xb=1 \quad .$$

This is a contradiction because it requires that 1 is a number in the neighborhood of infinity. If $x\in\mathbb{R}_\infty$ is a positive number less than every positive $\widehat{\mathbb{R}}$ number then we find that the product of two real numbers larger than one is equal to one. This is another contradiction. If x is a negative number then its product with a positive real number must be negative.

Remark 4.16 Since real numbers in the neighborhood infinity do not always have a multiplicative inverse, such numbers cannot be elements of number fields. The common practice of using number fields as a generalized proxy for all numbers, therefore, should be considered to have a narrower scope of valid application than is commonly understood. Also note that operations of the form $\widehat{\mathbb{R}} + \mathbb{R}$ are not closed because

$$\left(\widehat{\infty} - b\right) + a = \widehat{\infty} \quad , \qquad \text{for} \qquad b = a \quad , \quad b, a \in \mathbb{R}_0 \quad ,$$

and

$$\widehat{\infty} \notin \widehat{\mathbb{R}} \quad .$$

§5 Complex Numbers

Definition 5.1 A number is a complex number $z \in \mathbb{C}$ if and only if

$$z = x + iy \quad , \qquad \text{where} \qquad x, y \in \mathbb{R} \quad , \quad i = \sqrt{-1} \quad .$$

Definition 5.2 As ∞ does not absorb -1 in 1D, in 2D we have the condition that infinity does not absorb -1 or $\pm i$.

Definition 5.3 As the extended real line has two distinct infinites, the extended complex plane has four: $\{+\infty, +i\infty, -\infty, -i\infty\}$.

Definition 5.4 The affinely extended complex plane is

$$\overline{\mathbb{C}} \quad \equiv \quad \mathbb{C} \cup \{\pm\infty\} \cup \{\pm i\infty\} \quad .$$

Definition 5.5 The multiplicative operations for $\pm\infty$ and $\pm i\infty$ with i are

$$\begin{aligned} \pm\infty \cdot i = i \cdot \pm\infty &= \pm i\infty \\ \pm i\infty \cdot i = i \cdot \pm i\infty &= \mp\infty \quad . \end{aligned}$$

Remark 5.6 The non-distributive property of $\pm\infty$ (Definition 2.8) was practically redundant in 1D but for $z \in \mathbb{C}$ this feature gains significance.

Definition 5.7 The multiplicative operations for $\pm\infty$ with complex numbers $z \in \mathbb{C}$ are

$$\pm\infty \cdot z = z \cdot \pm\infty = \begin{cases} \pm\infty & \text{if} \quad \mathrm{Re}(z) > 0 \quad \text{and} \quad \mathrm{Im}(z) = 0 \\ \mp\infty & \text{if} \quad \mathrm{Re}(z) < 0 \quad \text{and} \quad \mathrm{Im}(z) = 0 \\ \pm i\infty & \text{if} \quad \mathrm{Im}(z) > 0 \quad \text{and} \quad \mathrm{Re}(z) = 0 \\ \mp i\infty & \text{if} \quad \mathrm{Im}(z) < 0 \quad \text{and} \quad \mathrm{Re}(z) = 0 \\ \text{undefined} & \text{if} \quad \mathrm{Im}(z) \neq 0 \quad \text{and} \quad \mathrm{Re}(z) \neq 0 \\ \text{undefined} & \text{if} \quad z = 0 \end{cases} \quad .$$

Definition 5.8 The multiplicative operations for $\pm i\infty$ with complex numbers $z \in \mathbb{C}$ are

$$\pm i\infty \cdot z = z \cdot \pm i\infty = \begin{cases} \pm i\infty & \text{if} \quad \mathrm{Re}(z) > 0 \quad \text{and} \quad \mathrm{Im}(z) = 0 \\ \mp i\infty & \text{if} \quad \mathrm{Re}(z) < 0 \quad \text{and} \quad \mathrm{Im}(z) = 0 \\ \mp\infty & \text{if} \quad \mathrm{Im}(z) > 0 \quad \text{and} \quad \mathrm{Re}(z) = 0 \\ \pm\infty & \text{if} \quad \mathrm{Im}(z) < 0 \quad \text{and} \quad \mathrm{Re}(z) = 0 \\ \text{undefined} & \text{if} \quad \mathrm{Im}(z) \neq 0 \quad \text{and} \quad \mathrm{Re}(z) \neq 0 \\ \text{undefined} & \text{if} \quad z = 0 \end{cases} .$$

Definition 5.9 The arithmetic operations for complex numbers $z \in \mathbb{C}$ whose real and/or imaginary parts are $\widehat{\mathbb{R}}$ numbers follow directly from the other definitions.

§6 The Riemann Zeta Function

Example 6.1 The Euler product is constructed from the Dirichlet sum form of ζ as follows. We expand the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \frac{1}{5^z} + \dots \quad ,$$

and then multiply both sides by 2^{-z} to obtain

$$\frac{1}{2^z}\zeta(z) = \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \frac{1}{8^z} + \frac{1}{10^z} + \dots \quad .$$

The next step is to subtract the second equation from the first to obtain

$$\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{9^z} + \frac{1}{11^z} + \dots \quad .$$

Then we multiply by 3^{-z} to obtain

$$\frac{1}{3^z}\left(1 - \frac{1}{2^z}\right)\zeta(z) = \frac{1}{3^z} + \frac{1}{9^z} + \frac{1}{15^z} + \frac{1}{21^z} + \frac{1}{27^z} + \dots \quad .$$

Subtracting again yields

$$\left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{11^z} + \dots \quad .$$

Next, we would multiply by 5^{-z} and then subtract again. Essentially, we apply the Sieve of Eratosthenes to get rid of all the fractional terms on the right. The issue of absolute converges arises when we note that even in the limit of sieving out an infinite number of primes, there might still be another term left on the right hand side after the 1. In the limit, that number is infinitely large, and for $z > 1$ the final term is guaranteed to be zero through $1/\infty = 0$. For this reason, we say ζ absolutely converges to the Euler product on the region of the complex plane where $\mathrm{Re}(z) > 1$. If we took, for example, $z = -2$, then remaining term on the right would be ∞ and we could not nicely divide the parenthetical terms from the left to obtain

$$\zeta(z) = \frac{1}{\left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{5^z}\right)\left(1-\frac{1}{7^z}\right)\dots} = \prod_{p\,|\,\mathrm{primes}} \frac{1}{1-p^{-z}} \quad .$$

Definition 6.2 An application of the Riemann ζ function is said to be an application in quantum field theory if and only if ζ appears inside the complex exponential as

$$\psi := e^{i\zeta(z)} \quad .$$

Theorem 6.3 *For any application in quantum field theory, the Riemann ζ function absolutely converges to the Euler product if z is such that $Re(z) < -1$.*

Proof. To prove this theorem, we will construct the Euler product inside the complex exponential. We have

$$e^{i\zeta(z)} = e^{i\left(1+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\frac{1}{5^z}+\dots\right)} \quad .$$

Multiply this by $e^{i(2^{-z})}$ to obtain

$$e^{i\frac{1}{2^z}\zeta(z)} = e^{i\left(\frac{1}{2^z}+\frac{1}{4^z}+\frac{1}{6^z}+\frac{1}{8^z}+\dots\right)} \quad .$$

Next, divide the first equation by the second to obtain

$$e^{i\left(1-\frac{1}{2^z}\right)\zeta(z)} = e^{i\left(1+\frac{1}{3^z}+\frac{1}{5^z}+\frac{1}{7^z}+\frac{1}{9^z}+\frac{1}{11^z}+\dots\right)} \quad .$$

By following the prescription put forth in Example 6.1, we obtain

$$e^{i\left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{5^z}\right)\dots\ \zeta(z)} = e^{i(1+\dots)} \quad .$$

As in Example 6.1, if $\mathrm{Re}(z) > 1$ the last term in the exponent is guaranteed to go to zero and we have

$$e^{i\zeta(z)} = e^{\frac{i}{\left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{5^z}\right)\dots}} = e^{i\prod\frac{1}{1-p^{-z}}} \quad .$$

If $\mathrm{Re}(z) < -1$, then we obtain

$$e^{i\left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{5^z}\right)\dots\ \zeta(z)} = e^{i(1+\widehat{\infty})} = e^{i}e^{i\widehat{\infty}} \quad .$$

We have proven in References [1] and [2] that $e^{i\widehat{\infty}}=1$. Therefore, when $\text{Re}(z)<-1$, we obtain the absolutely convergent expression

$$e^{i\left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{5^z}\right)\dots\,\zeta(z)}=e^{i(1+\widehat{\infty})}=e^ie^{i\widehat{\infty}}=e^{i\prod\frac{1}{1-p^{-z}}}\quad .$$

We have proven that ζ does absolutely converge to the Euler product for every application in quantum field theory when $\text{Re}(z)<-1$.

Theorem 6.4 *The Euler product form of the Riemann zeta function has complex zeros in the neighborhood of negative infinity.*

Proof. Consider a number $z_0\in\mathbb{C}$ such that

$$z_0=-\big(\widehat{\infty}-b\big)+iy_0\quad,\qquad\text{where}\qquad b,y_0\in\mathbb{R}_0\quad,\quad b\neq0\quad .$$

Observe that the Euler product form of ζ

$$\zeta(z)=\prod_{p|primes}\frac{1}{1-p^{-z}}$$

takes z_0 as

$$\begin{aligned}\zeta(z_0)&=\frac{1}{1-P^{(\widehat{\infty}-b)-iy_0}}\left(\prod_{\substack{p|primes\\p\neq P}}\frac{1}{1-p^{(\widehat{\infty}-b)-iy_0}}\right)\\&=\frac{1}{1-\dfrac{1}{P^b}P^{\widehat{\infty}}\big[\cos(y_0\ln P)-i\sin(y_0\ln P)\big]}\left(\prod_{\substack{p|primes\\p\neq P}}\frac{1}{1-p^{(\widehat{\infty}-b)-iy_0}}\right)\quad .\end{aligned}$$

Let $y_0\ln P=2n\pi$ for some prime P and $n\in\mathbb{N}$. Then

$$\zeta(z_0)=\frac{1}{1-\widehat{\infty}}\left(\prod_{\substack{p|primes\\p\neq p'}}\frac{1}{1-p^{(\widehat{\infty}-b)-iy_0}}\right)\quad .$$

By Theorem 4.8, we find that

$$\zeta(z_0)=0\quad .$$

Remark 6.5 Theorem 6.4 demonstrates that the Riemann hypothesis is effectively false for all applications in quantum field theory.

References

[1] Jonathan W. Tooker. Proof of the Limits of Sine and Cosine at Infinity. *viXra:1809.0234*, (2018).

[2] Jonathan W. Tooker. Derivation of the Limits of Sine and Cosine at Infinity. *viXra:1806.0082*, (2018).