The Lorentz Upshot of the Galilean Group's Fundamentals

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Abstract

The fundamental properties of the (x,t) Galilean inertial transformations include their homogeneous linearity, their intrinsic velocity v, where setting v to zero produces the identity transformation and negation of v inverts the transformation, and their closure under composition. We show that stipulation of these three fundamental (x,t) Galilean inertial transformation properties yields all generic (x,t) Lorentz transformation groups, which are distinguished by their speed constant values that supplant c; the (x,t) Galilean group itself is the generic (x,t) Lorentz group with infinite speed constant.

Introduction

The (x,t) Galilean inertial transformations, which are given by,

$$x' = x - vt,\ t' = t, \tag{1a}$$

are obviously *homogeneously linear*, and they furthermore of course imply that,

$$dx' = dx - vdt,\ dt' = dt, \tag{1b}$$

so they *in addition* yield the *corresponding* (dx/dt) Galilean inertial *velocity* transformations,

$$(dx'/dt') = (dx/dt) - v. \tag{1c}$$

Since $(dx'/dt') = -v$ when $(dx/dt) = 0$, v is a Galilean inertial transformation's *intrinsic velocity*. When *its intrinsic velocity v vanishes*, a Galilean inertial transformation *reduces to the identity transformation*, i.e.,

$$v = 0 \text{ implies that } x' = x,\ t' = t \text{ and } (dx'/dt') = (dx/dt). \tag{1d}$$

The *inverse* of a Galilean inertial transformation is easily worked out to be,

$$x = x' + vt',\ t = t' \text{ and } (dx/dt) = (dx'/dt') + v, \tag{1e}$$

so a Galilean inertial transformation's *inversion* is achieved simply by $v \to -v$.

The *composition* of a Galilean inertial transformation of intrinsic velocity v_1 with another of intrinsic velocity v_2 *is as well a Galilean inertial transformation*, namely *one whose intrinsic velocity is* $(v_1 + v_2)$. That fact is readily demonstrated by writing Galilean inertial transformations *in matrix notation*, i.e., as,

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} x - vt \\ t \end{pmatrix} = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}, \tag{1f}$$

and then calculating *the composition of two Galilean inertial transformations by using matrix multiplication*,

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \begin{pmatrix} 1 & -v_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -v_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 & -(v_1 + v_2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}. \tag{1g}$$

The above discussion has highlighted *three fundamental properties* of the (x,t) Galilean inertial transformations: 1) their *homogeneous linearity*, 2) their *intrinsic velocity* v, where $v = 0$ *produces the identity transformation* and $v \to -v$ *inverts the transformation* and 3) their *closure under composition*. In the next section we show that stipulation of these three fundamental (x,t) Galilean inertial transformation properties *yields all generic* (x,t) *Lorentz transformation groups*, which are distinguished by their speed constant values that supplant c; the (x,t) Galilean group is the generic (x,t) Lorentz group with *infinite* speed constant.

The three Galilean-group fundamentals which yield all generic Lorentz groups

The most *general* possible *homogeneously linear* (x,t) *transformations* have the form,

$$x' = \alpha x - \beta(vt),\ t' = \gamma t - \delta(x/v), \tag{2a}$$

where α, β, γ and δ are dimensionless. These transformations of course imply that,

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$$dx' = \alpha dx - \beta(vdt),\ dt' = \gamma dt - \delta(dx/v), \tag{2b}$$

so they *also* imply the *corresponding* rational (dx/dt) *velocity* transformations,

$$(dx'/dt') = ((\alpha/\gamma)(dx/dt) - (\beta/\gamma)v)/(1 - (\delta/(\gamma v))(dx/dt)). \tag{2c}$$

Since $(dx'/dt') = -(\beta/\gamma)v$ when $(dx/dt) = 0$, v is the Eq. (2a) transformation's *intrinsic velocity* only if,

$$\beta = \gamma. \tag{2d}$$

With the Eq. (2d) constraint, the Eq. (2c) *velocity transformation* simplifies to,

$$(dx'/dt') = ((\alpha/\gamma)(dx/dt) - v)/(1 - (\delta/(\gamma v))(dx/dt)). \tag{2e}$$

The *inverse* of the Eq. (2e) *velocity transformation* is straightforwardly worked out to be,

$$(dx/dt) = ((dx'/dt') + v)/((\alpha/\gamma) + (\delta/(\gamma v))(dx'/dt')). \tag{2f}$$

Eqs. (2f) and (2e) only fulfill the condition *that transformation inversion is achieved by* $v \to -v$ if,

$$\alpha = \gamma \text{ and } (\delta/\gamma) \textit{ is an even function of } v. \tag{2g}$$

With the Eq. (2g) constraint, the Eq. (2e) *velocity transformation* simplifies to,

$$(dx'/dt') = ((dx/dt) - v)/(1 - (\delta/(\gamma v))(dx/dt)),\ \text{where } (\delta/\gamma) \textit{ is an even function of } v. \tag{2h}$$

With the constraints given by Eqs. (2d) and (2g), the Eq. (2a) (x, t) *transformation* simplifies to,

$$x' = \gamma(x - vt),\ t' = \gamma(t - (\delta/(\gamma v))x), \tag{2i}$$

which, when *inverted*, reads,

$$x = (x' + vt')/(\gamma(1 - (\delta/\gamma))),\ t = (t' + (\delta/(\gamma v))x')/(\gamma(1 - (\delta/\gamma))). \tag{2j}$$

Eqs. (2j) and (2i) only fulfill the condition *that transformation inversion is achieved by* $v \to -v$ if,

$$(\delta/\gamma) = 1 - \gamma^{-2} \text{ and } \gamma \textit{ is an even function of } v. \tag{2k}$$

The constraints given by Eq. (2k) imply that the (x, t) transformation given by Eq. (2i) simplifies to,

$$x' = \gamma(x - vt),\ t' = \gamma\left(t - \left(1 - \gamma^{-2}\right)(x/v)\right),\ \text{where } \gamma \textit{ is an even function of } v, \tag{2l}$$

and the Eq. (2k) constraints of course as well imply that the Eq. (2h) *velocity transformation* simplifies to,

$$(dx'/dt') = ((dx/dt) - v)/\left(1 - \left(1 - \gamma^{-2}\right)((dx/dt)/v)\right),\ \text{where } \gamma \textit{ is an even function of } v. \tag{2m}$$

Taking note at this point of the fact that *the insertion of* $v = 0$ into Eqs. (2l) and (2m) must produce the respective *identity* transformations, it is clear that *not only must* γ *be an even function of* v*, its asymptotic behavior for vanishingly small* v *also must be*,

$$\gamma \sim 1 + O(v^2) \text{ as } v \to 0. \tag{2n}$$

The *weak constraints on the behavior of* γ inherent in Eq. (2n) and in the fact that γ must be an even function of v turn out to be *greatly strengthened* by the requirement that the *composition* of two (x, t) transformations of the form given by Eq. (2l) *must itself be an* (x, t) *transformation of that* Eq. (2l) *form.* To enforce this *composition closure* requirement for the Eq. (2l) transformations, we first *write them in matrix notation*, in analogy to what was done in Eq. (1f) for the specifically Galilean inertial transformations,

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma(x - vt) \\ \gamma\left(t - \left(1 - \gamma^{-2}\right)(x/v)\right) \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma\left(1 - \gamma^{-2}\right)v^{-1} & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}, \tag{2o}$$

and then calculate the *composition of two such* Eq. (2o) *matrix transformations by using matrix multiplication*, which we *follow* by requiring *that result to itself adhere to the specific* Eq. (2o) *matrix form*,

$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\gamma_2 v_2 \\ -\gamma_2\left(1-\gamma_2^{-2}\right)v_2^{-1} & \gamma_2 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix} =$$

$$\begin{pmatrix} \gamma_2 & -\gamma_2 v_2 \\ -\gamma_2\left(1-\gamma_2^{-2}\right)v_2^{-1} & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\gamma_1 v_1 \\ -\gamma_1\left(1-\gamma_1^{-2}\right)v_1^{-1} & \gamma_1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} =$$

$$\begin{pmatrix} \gamma_1\gamma_2\left(1+\left(1-\gamma_1^{-2}\right)v_1^{-1}v_2\right) & -\gamma_1\gamma_2(v_1+v_2) \\ -\gamma_1\gamma_2\left(\left(1-\gamma_1^{-2}\right)v_1^{-1}+\left(1-\gamma_2^{-2}\right)v_2^{-1}\right) & \gamma_1\gamma_2\left(1+\left(1-\gamma_2^{-2}\right)v_2^{-1}v_1\right) \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \tag{2p}$$

$$\begin{pmatrix} \gamma_{12} & -\gamma_{12} v_{12} \\ -\gamma_{12}\left(1-\gamma_{12}^{-2}\right)v_{12}^{-1} & \gamma_{12} \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}.$$

The last Eq. (2p) equality *constrains the composition of two* Eq. (2o) *matrix transformations to itself adhere to the specific* Eq. (2o) *matrix form*. One crucial consequence of the last Eq. (2p) equality is that,

$$\gamma_1\gamma_2\left(1+\left(1-\gamma_1^{-2}\right)v_1^{-1}v_2\right) = \gamma_{12} = \gamma_1\gamma_2\left(1+\left(1-\gamma_2^{-2}\right)v_2^{-1}v_1\right), \tag{2q}$$

which implies that,

$$\left(1-\gamma_1^{-2}\right)v_1^{-2} = \left(1-\gamma_2^{-2}\right)v_2^{-2}. \tag{2r}$$

Since *the values of* γ_1 *and* v_1 *are entirely independent of the values of* γ_2 *and* v_2, it must be the case that,

$$\left(1-\gamma_1^{-2}\right)v_1^{-2} = (v_0)^{-2} = \left(1-\gamma_2^{-2}\right)v_2^{-2}, \tag{2s}$$

where v_0 *is a constant with the dimension of speed*, which *we restrict to be either infinite or finite and positive*. Eq. (2s) implies that the Eq. (2l) parameter γ is related to the Eq. (2l) intrinsic velocity v by,

$$\left(1-\gamma^{-2}\right) = (v/v_0)^2, \tag{2t}$$

which, *when* $|v| < v_0$, *has the viable consequence that*

$$\gamma = \left(1/\sqrt{1-(v/v_0)^2}\right), \tag{2u}$$

where Eq. (2n) *requires the square root's positive sign*. If v_0 is infinite, Eq. (2u) *fixes* γ *to unity*, which causes Eq. (2l) *to describe the* (x, t) *Galilean transformation group*, but for finite positive v_0, the consequent γ of Eq. (2u) causes Eq. (2l) *to describe the generic* (x, t) *Lorentz transformation group in which* v_0 *supplants* c.

Moreover, insertion of Eq. (2t) into Eq. (2q) yields that,

$$\gamma_{12} = \gamma_1\gamma_2\left(1+\left(v_1v_2/v_0^2\right)\right), \tag{2v}$$

and, *in addition to* Eq. (2q), the last Eq. (2p) equality *furthermore implies the equation*,

$$\gamma_{12}v_{12} = \gamma_1\gamma_2(v_1+v_2), \tag{2w}$$

which, when divided by Eq. (2v), yields the generic (x, t) Lorentz group *intrinsic velocity composition rule*,

$$v_{12} = (v_1+v_2)/\left(1+\left(v_1v_2/v_0^2\right)\right). \tag{2x}$$

Eq. (2u) of course implies that γ_{12} is given by,

$$\gamma_{12} = \left(1/\sqrt{1-(v_{12}/v_0)^2}\right), \tag{2y}$$

which, by utilizing Eqs. (2x) and (2u), is readily shown *to be consistent with* Eq. (2v). In fact Eqs. (2x) and (2u) *are equivalent to all four of the equations which follow from the last* Eq. (2p) *equality*. As a consistency check, note that Eqs. (2x), (2y) and (2u) yield the correct Galilean results when $v_0 \to +\infty$.