A Simple Proof That Finite Mathematics Is More Fundamental Than Classical One

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Abstract

Classical mathematics (involving such notions as infinitely small/large and continuity) is usually treated as fundamental while finite mathematics is treated as inferior which is used only in special applications. In our previous publications we argue that the situation is the opposite: classical mathematics is only a special degenerate case of finite one in the formal limit when the characteristic of the ring or field in finite mathematics goes to infinity. In the present paper we give a simple and rigorous proof of this fundamental fact. In general, introducing infinity automatically implies transition to a degenerate theory because in that case all operations modulo a number are lost. So, *even from the pure mathematical point of view*, the very notion of infinity cannot be fundamental, and theories involving infinities can be only approximations to more general theories. We also prove that standard quantum theory based on classical mathematics is a special degenerate case of quantum theory based on finite mathematics. Motivation and implications are discussed.

Keywords: classical mathematics, finite mathematics, quantum theory

1 Remarks on arithmetic

The goal of the present paper is to give a mathematical proof that finite mathematics is more fundamental than classical one. At the same time, we believe that to better understand the problem it is important to discuss philosophical aspects of such a simple problem as operations with natural numbers.

In the 20s of the 20th century the Viennese circle of philosophers under the leadership of Schlick developed an approach called logical positivism which contains verification principle: *A proposition is only cognitively meaningful if it can be definitively and conclusively determined to be either true or false* (see e.g. Refs. [1]). On the other hand, as noted by Grayling [2], *"The general laws of science are not, even in principle, verifiable, if verifying means furnishing conclusive proof of their truth. They can be strongly supported by repeated experiments and accumulated evidence but they cannot be verified completely"*. Popper proposed the concept of falsificationism [3]: *If no cases where a claim is false can be found, then the hypothesis is accepted as provisionally true.*

According to the philosophy of quantum theory, there should be no statements accepted without proof and based on belief in their correctness (i.e. axioms). The theory should contain only those statements that can be verified, where by "verified" physicists mean experiments involving only a finite number of steps. So the philosophy of quantum theory is similar to verificationism, not falsificationism. Note that Popper was a strong opponent of the philosophy of quantum theory and supported Einstein in his dispute with Bohr.

The verification principle does not work in standard classical mathematics. For example, it cannot be determined whether the statement that $a+b=b+a$ for all natural numbers a and b is true or false. According to falsificationism, this statement is provisionally true until one has found some numbers a and b for which $a+b \neq b+a$. There exist different theories of arithmetic (e.g. Peano arithmetic or Robinson arithmetic) aiming to solve foundational problems of standard arithmetic. However, those theories are incomplete and are not used in applications.

From the point of view of verificationism and the philosophy of quantum theory, classical mathematics is not well defined not only because it contains an infinite number of numbers. For example, let us pose a problem whether 10+20 equals 30. Then one should describe an experiment which gives the answer to this problem. Any computing device can operate only with a finite number of bits and can perform calculations only modulo some number p. Say $p=40$, then the experiment will confirm that 10+20=30 while if $p=25$ then we will get that 10+20=5.

So the statements that 10+20=30 and even that $2\cdot 2=4$ are ambiguous because they do not contain information on how they should be verified. On the other hands, the statements

$$10+20=30\,(mod\,40),\;\;10+20=5\,(mod\,25),$$

$$2\cdot 2=4\,(mod\,5),\;\;2\cdot 2=2\,(mod\,2)$$

are well defined because they do contain such an information. So, from the point of view of verificationism and the philosophy of quantum theory, only operations modulo a number are well defined.

We believe the following observation is very important: although classical mathematics (including its constructive version) is a part of our everyday life, people typically do not realize that *classical mathematics is implicitly based on the assumption that one can have any desired amount of resources.* So classical mathematics is based on the implicit assumption that we can consider an idealized case when a computing device can operate with an infinite number of bits. In other words, standard operations with natural numbers are implicitly treated as limits of operations modulo p when $p\to\infty$. Usually in mathematics, legitimacy of every limit is thoroughly investigated, but in the simplest case of standard operations with natural numbers it is not even mentioned that those operations can be treated as limits of operations modulo p. In real life such limits even might not exist if, for example, the Universe contains a finite number of elementary particles.

Classical mathematics proceeds from standard arithmetic which does not contain operations modulo a number while finite mathematics necessarily involves such operations. In the present paper we prove that, regardless of philosophical preferences, finite mathematics is more fundamental than classical one. In Sec. 2 we define a criterion when one theory is more fundamental than the other, and the proof of the main statement is given in Sec. 3.

2 Comparison of different theories

A belief of the overwhelming majority of scientists is that classical mathematics (involving the notions of infinitely small/large and continuity) is fundamental while finite mathematics is something inferior what is used only in special applications. This belief is based on the fact that the history of mankind undoubtedly shows that classical mathematics has demonstrated its power in many areas of science.

The notions of infinitely small/large, continuity etc. were proposed by Newton and Leibniz more than 300 years ago. At that time people did not know about existence of atoms and elementary particles and believed that any body can be divided by an arbitrarily large number of arbitrarily small parts. However, now it is obvious that standard division has only a limited applicability because when we reach the level of atoms and elementary particle the division operation looses its meaning. In nature there are no infinitely small objects and no continuity because on the very fundamental level nature is discrete. So, as far as application of mathematics to physics is concerned, classical mathematics is only an approximation which in many cases works with very high accuracy but the ultimate quantum theory cannot be based on classical mathematics.

A typical situation in physics can be described by the following

Definition: *Let theory A contain a finite parameter and theory B be obtained from theory A in the formal limit when the parameter goes to zero or infinity. Suppose that with any desired accuracy theory A can reproduce any result of theory B by choosing a value of the parameter. On the contrary, when the limit is already taken then one cannot return back to theory A and theory B cannot reproduce all results of theory A. Then theory A is more general than theory B and theory B is a special degenerate case of theory A.*

Probably the most known example is that nonrelativistic theory (NT) can be obtained from relativistic theory (RT) in the formal limit $c \to \infty$ where c is the velocity of light. RT can reproduce any result of NT with any desired accuracy if c is chosen to be sufficiently large. On the contrary, when the limit is already taken then one cannot return back from NT to RT, and NT can reproduce results of RT only in relatively small amount of cases when velocities are much less than c. Therefore RT is more general than NT, and NT is a special degenerate case of RT. Other known examples are that classical theory is a special degenerate case of quantum one in the formal limit $\hbar \to 0$ where $\hbar$ is the Planck constant, and Poincare invariant theory is a special degenerate case of de Sitter invariant theories in the formal limit $R \to \infty$

where R is the parameter defining contraction from the de Sitter Lie algebras to the Poincare Lie algebra.

In our publications (see e.g. Refs. [4, 5]) we discussed an approach called Finite Quantum Theory (FQT) where quantum theory is based not on classical but on finite mathematics. It has been shown that FQT is more general than standard quantum theory and the latter is a special degenerate case of the former in the formal limit when the characteristic of the field or ring in FQT goes to infinity. In Sec. 3 we describe a proof of this statement and also prove

Main Statement: Even classical mathematics itself is a special degenerate case of finite mathematics in the formal limit when the characteristic of the field or ring in the latter goes to infinity.

Note that this statement is meaningful only if in applications finite mathematics is more pertinent than classical one while if those theories are treated only as abstract ones than the statement that one theory is more fundamental than the other is meaningless.

3 Proof of the main statement

Classical mathematics starts from natural numbers but here only addition and multiplication are always possible. In order to make addition invertible we introduce negative integers and get the ring of integers Z. However, if instead of all natural numbers we consider only a set R_p of p numbers 0, 1, 2, ... $p-1$ where addition and multiplication are defined as usual but modulo p then we get a ring without adding new elements. In our opinion the notation Z/p for R_p is not quite adequate because it may give a wrong impression that finite mathematics starts from the infinite set Z and that Z is more general than R_p. However, although Z has more elements than R_p, Z cannot be more general than R_p because Z does not contain operations modulo a number.

Since operations in R_p are modulo p, one can represent R_p as a set $\{0, \pm 1, \pm 2, ..., \pm(p-1)/2)\}$ if p is odd and as a set $\{0, \pm 1, \pm 2, ..., \pm(p/2-1), p/2\}$ if p is even. Let f be a function from R_p to Z such that $f(a)$ has the same notation in Z as a in R_p. If elements of Z are depicted as integer points on the x axis of the xy plane then, if p is odd, the elements of R_p can be depicted as points of the circumference in Fig. 1. and analogously if p is even. This picture is natural since R_p has a property that if we take any element $a \in R_p$ and sequentially add 1 then after p steps we will exhaust the whole set R_p by analogy with the property that if we move along a circumference in the same direction then sooner or later we will arrive at the initial point.

We define the function $h(p)$ such that $h(p) = (p-1)/2$ if p is odd and $h(p) = p/2 - 1$ if p is even. Let n be a natural number and $U(n)$ be a set of elements $a \in R_p$ such that $|f(a)|^n \leq h(p)$. Then $\forall m \leq n$ the result of any m operations of addition, subtraction or multiplication of elements $a \in U(n)$ is the same as for the corresponding elements $f(a)$ in Z, i.e. in this case operations modulo p are not

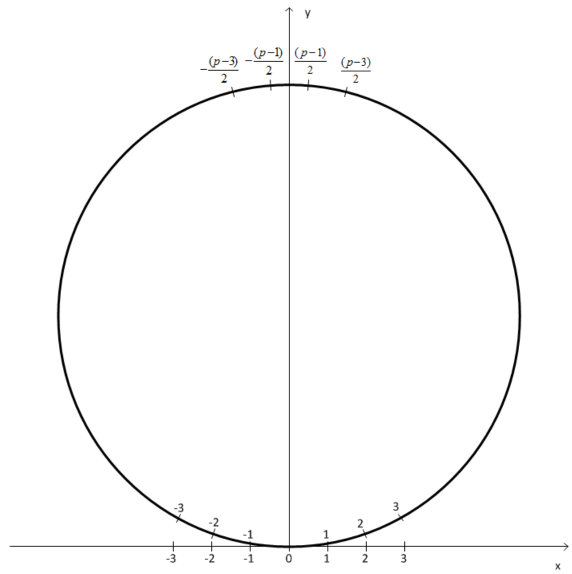

Figure 1: Relation between R_p and Z

explicitly manifested.

Let $n = g(p)$ be a function of p and $G(p)$ be a function such that the set $U(g(p))$ contains at least the elements $\{0, \pm 1, \pm 2, ..., \pm G(p)\}$. In what follows $M > 0$ is a natural number. If there is a sequence of natural numbers (a_n) then standard definition that $(a_n) \to \infty$ is that $\forall M\ \exists N > 0$ such that $a_n \geq M\ \forall n \geq N$. By analogy with this definition we will now prove

Statement 1: There exist functions $g(p)$ and $G(p)$ such that $\forall M\ \exists p_0 > 0$ such that $g(p) \geq M$ and $G(p) \geq 2^M\ \forall p \geq p_0$.

Proof. $\forall p > 0$ there exists a unique natural n such that $2^{n^2} \leq h(p) < 2^{(n+1)^2}$. Define $g(p) = n$ and $G(p) = 2^n$. Then $\forall M\ \exists p_0$ such that $h(p_0) \geq 2^{M^2}$. Then $\forall p \geq p_0$ the conditions of *Statement 1* are satisfied. ∎

The problem of actual infinity is discussed in a vast literature. The *technique* of classical mathematics does not involve actual infinities and here infinities are understood only as limits. However, the *basis* of classical mathematics does involve actual infinities. For example, Z is treated as actual and not potential infinity and there is no rigorous definition of Z as a limit of some finite set. *Statement 1* is the proof that the ring Z is the limit of the ring R_p when $p \to \infty$, and the result of any finite combination of additions, subtractions and multiplications in Z can be reproduced in R_p if p is chosen to be sufficiently large. On the contrary, when the limit is already taken then one cannot return back from Z to R_p, and in Z it is not possible to reproduce all results in R_p because in Z there are no operations modulo a number. According to **Definition** in Sec. 2 this means that *the ring R_p is more general than Z, and Z is a special degenerate case of R_p.*

When p is very large then $U(g(p))$ is a relatively small part of R_p, and in general the results in Z and R_p are the same only in $U(g(p))$. This is analogous to the fact mentioned in Sec. 2 that the results of NT and RT are the same only in relatively small cases when velocities are much less than c. However, when the radius of the circumference in Fig. 1 becomes infinitely large then a relatively small vicinity of zero in R_p becomes the infinite set Z when $p \to \infty$. *This example demonstrates that once we involve infinity and replace R_p by Z then we automatically obtain a degenerate theory because in Z there are no operations modulo a number.*

In classical mathematics the ring Z is the starting point for introducing the notions of rational and real numbers. Therefore those notions arise from a degenerate set. Then a question arises whether the fact that R_p is more general than Z implies that finite mathematics is more general than classical one, i.e. in particular whether finite mathematics can reproduce the results obtained by applications of classical mathematics. For example, if p is prime then R_p becomes the Galois field F_p, and the results in F_p considerably differ from those in the set Q of rational numbers even when p is very large. In particular, $1/2$ in F_p is a very large number $(p+1)/2$. Since quantum theory is the most general physical theory, the answer to this question depends on whether standard quantum theory based on classical mathematics is most general or is a special degenerate case of a more general quantum theory.

As noted in Sec. 2, de Sitter invariant quantum theory is more general than Poincare invariant quantum theory. As shown in Refs. [4, 5]

Statement 2: In standard de Sitter invariant quantum theory it is always possible to find a basis where the spectrum of all operators is purely discrete and the eigenvalues of those operators are elements of Z. Therefore the remaining problem is whether or not quantum theory based on finite mathematics can be a generalization of standard quantum theory where states are described by elements of a separable complex Hilbert spaces H.

Let x be an element of H and $(e_1, e_2, ...)$ be a basis of H normalized such that the norm of each e_j is an integer. Then *with any desired accuracy each element of H can be approximated by a finite linear combination*

$$x = \sum_{j=1}^{n} c_j e_j \qquad (1)$$

where $c_j = a_j + ib_j$ and all the numbers a_j and b_j $(j = 1, 2, ...n)$ are rational. This follows from the known fact that the set of such sums is dense in H.

The next observation is that spaces in quantum theory are projective, i.e. for any complex number c the elements x and cx describe the same state. This follows from the physical fact that not the probability itself but only ratios of probabilities have a physical meaning. In view of this property, both parts of Eq. (1) can be multiplied by a common denominator of all the numbers a_j and b_j. As a result, we have

Statement 3: Each element of H can be approximated by a finite linear combination (1) where now all the numbers a_j and b_j are integers, i.e. belong to Z.

We conclude that Hilbert spaces in standard quantum theory contain a big redundancy of elements. Indeed, although formally the description of states in standard quantum theory involves rational and real numbers, such numbers play only an auxiliary role because with any desired accuracy each state can be described by using only integers. Therefore, as follows from **Definition** in Sec. 2 and *Statements 1-3*,

- Standard quantum theory based on classical mathematics is a special degenerate case of quantum theory based on finite mathematics.
- **Main Statement** formulated in Sec. 2 is valid.

4 Discussion

The above construction has a well-known historical analogy. For many years people believed that the Earth was flat and infinite, and only after a long period of time they realized that it was finite and curved. It is difficult to notice the curvature dealing only with distances much less than the radius of the curvature. Analogously one might think that the set of numbers describing physics in our Universe has a "curvature" defined by a very large number p but we do not notice it dealing only with numbers much less than p.

As noted in the preceding section, introducing infinity automatically implies transition to a degenerate theory because in this case operations modulo a number are lost. Therefore *even from the pure mathematical point of view* the notion of infinity cannot be fundamental, and theories involving infinities can be only approximations of more general theories.

In the preceding sections we have proved that classical mathematics is a special degenerate case of finite one in the formal limit $p \to \infty$ and that quantum theory based on finite mathematics is more fundamental than standard quantum theory. The fact that at the present stage of the Universe p is a huge number explains why in many cases classical mathematics describes natural phenomena with a very high accuracy. At the same time, as shown in Ref. [5], the explanation of several phenomena can be given only in the theory where p is finite.

One of the examples is that in our approach gravity is a manifestation of the fact that p is finite. In Ref. [5] we derive the approximate expression for the gravitational constant wich depends on p as $1/lnp$. By comparing this expression with the experimental value we get that lnp is of the order of 10^{80} or more, i.e. p is a huge number of the order of $exp(10^{80})$ or more. However, since lnp is "only" of the order of 10^{80} or more, the existence of p is observable while in the formal limit $p \to \infty$ gravity disappears.

Although classical mathematics is a degenerate case of finite one, a problem arises whether classical mathematics can be substantiated as an abstract science. It is well-known that, in spite of great efforts of many great mathematicians, the problem of foundation of classical mathematics has not been solved. For example, Gödel's incompleteness theorems state that no system of axioms can ensure that all facts about natural numbers can be proven and the system of axioms in classical mathematics cannot demonstrate its own consistency. Let us recall that classical mathematics does not involve operations modulo a number.

The philosophy of Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and other great mathematicians was based on macroscopic experience in which the notions of infinitely small, infinitely large, continuity and standard division are natural. However, as noted above, those notions contradict the existence of elementary particles and are not natural in quantum theory. The illusion of continuity arises when one neglects the discrete structure of matter.

However, since in applications classical mathematics is a special degenerate case of finite one, foundational problems in this mathematics are important only when it is treated as an abstract science. The technique of classical mathematics is very powerful and in many cases (but not all of them) describes reality with a high accuracy.

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