An optimization approach to the Riemann Hypothesis

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Abstract. Optimization of relevant concepts such as action or utility functions enabled the derivation of theories and laws in some scientific fields such as physics and economics. This fact suggested that the problem of the location of the Riemann Zeta Function's (RZF) nontrivial zeros can be addressed in a mathematical programming framework. Using a constrained nonlinear optimization formulation of the problem, we prove that RZF's nontrivial zeros are located on the critical line, thereby confirming the Riemann Hypothesis. This result is a direct implication of the Kuhn-Tucker necessary optimality conditions for the formulated nonlinear program.

Keywords: Riemann Zeta function, Riemann Hypothesis, Optimization, Kuhn-Tucker conditions.

Introduction

A great deal of research has been and still is devoted to the zeros of the Riemann Zeta function (RZF) that are located in the critical strip[2] and known as the nontrivial zeros of RZF. The Riemann Hypothesis (RH) states that these zeros are located on the critical line[3]. Although a large number of nontrivial zeros have proved to be located on the critical line through numerical computation methods, starting with Riemann's manual computation of the first few zeros [1], no analytical proof or disproof of RH has been developed since its conjecture by Riemann in 1859.

In this paper, we propose an analytical approach to RH based on optimization. This tool proved successful in deriving some important scientific theories and laws [2]. By formulating and solving the appropriate optimization problem, we derive evidence in support of the Riemann Hypothesis.

Proof

We denote the Riemann zeta function (RZF) as $\zeta(\sigma+it) = U(\sigma,t)+iV(\sigma,t)$, for complex $s = \sigma+it$. As a consequence of the properties of RZF and the properties of its nontrivial zeros[4], the search for the location of these zeros can be limited to the half plane on the left of the critical line. Zeros on the right of the critical line can be obtained by symmetry about this line. Also RZF's functional equation shows that nontrivial zeros occur either on the critical line, or in pairs, off of the critical, that are symmetric about this line.

Hence, this search entails finding the value σ^* where $\zeta(\sigma^*+it)$ or equivalently $|\zeta(\sigma^*; t)|^2$, vanishes at some height $t = t^*$. In this framework this task can be accomplished by minimizing the simple objective function $|\zeta(\sigma; t^*)|^2$ under the constraint on $0 < \sigma \leq ½$, t^* being a constant.

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[2] *A strip in the complex plane defined by $0 \leq \sigma \leq 1$*

[3] *The critical line is the line in the complex plane defined by $\sigma = 1/2$*

[4] *See Appendix for details*

With a nonnegativity constraint on σ for nontrivial zeros, the optimization problem of interest is then:

Minimize $f(\sigma) = Z(\sigma; t^*) = |\zeta(\sigma; t^*)|^2 = U^2(\sigma; t^*) + V^2(\sigma; t^*)$

Subject to: $g(\sigma) = \sigma - 1/2 \leq 0$ (P)

$\sigma > 0,$

To solve the nonlinear constrained problem (P), we use the Kuhn-Tucker method [3].The Lagrange function associated with (P) is then:

$$L(\sigma, \mu) = Z(\sigma; t^*) + \mu(\sigma - ½)$$

Where μ is the Lagrange multiplier associated with the constraint $g(\sigma)$.

For minimization problems with continuously differentiable functions (f and g in problem P), nonnegative variables (σ in our case), and under a regularity qualification of the constraints[5], optimality at $\sigma = \sigma^*$ requires the existence of a Lagrange multiplier μ^* such that the following conditions are satisfied [4]:

1. Complementary slackness conditions

$$\sigma^* L_\sigma(\sigma^*, \mu^*) = \sigma^*(Z_\sigma^* + \mu^*) = 0 \quad (1)$$ [6]

$$\mu^* L_\mu(\sigma^*, \mu^*) = \mu^*(\sigma^* - ½) = 0 \quad (2)$$ [7]

2. Feasibility conditions

$$L_\sigma(\sigma^*, \mu^*) = Z_\sigma^* + \mu^* \geq 0 \quad (3)$$ [8]

$$L_\mu(\sigma^*, \mu^*) = (\sigma^* - ½) \leq 0 \quad (4)$$

$$\sigma^* > 0 \quad (5)$$

$$\mu^* \geq 0 \quad (6)$$

Being equality conditions, the complementary slackness conditions (1) and (2) provide the system of equations that is used to find potential candidate solutions to the mathematical program (P). Any candidate solution will have to meet the feasibility conditions as well as the properties of RZF and its nontrivial zeros to qualify as a candidate solution to problem (P).

Since feasibility condition (5) requires that $\sigma^* > 0$, condition (1) reduces to (1b):

$$Z^*_\sigma + \mu^* = 0 \quad (1b)$$

Moreover, $\zeta(\sigma; t^*)$ vanishing at $t = t^*$, for $\sigma = \sigma^*$ implies that:

$$Z^*_\sigma = U^*U^*_\sigma + V^*V^*_\sigma = 0$$ [9]

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[5] *The gradients of the equality and binding nonequality constraints have to be linearly independent at the stationary/critical point(s) of the Lagrangian function. This requirement is of no concern here since we have one constraint only.*

[6] $L_\sigma = \partial L / \partial\sigma$

[7] $L_\mu = \partial L / \partial\mu$

[8] $Z_\sigma = \partial Z / \partial\sigma$

[9] *Since RZF is analytic in the critical line, its real and its imaginary parts, U(s) and V(s) are differentiable in the critical strip hence:* $Z_\sigma^* = U^*U^*_\sigma + V^*V^*_\sigma = 0$

Hence, from (1b) we get $\mu^* = 0$ as a necessary value of μ^* at optimality. Feasibility conditions (1), (2, (3) and (6) are therefore satisfied.

Condition (4), expresses the property that nontrivial zeros occur either on the critical line (for $g(\sigma^*) = 0$), or in pairs, off of the critical line (for $g(\sigma^*) < 0$), that are symmetric about this line, for any t^* where RZF vanishes.

If $g(\sigma^*)$ is binding, that is $g(\sigma^*) = 0$, then $\sigma^* = ½$ and the corresponding zero is therefore on the critical line. The alternative is that $g(\sigma^*)$ is not binding, that is $g(\sigma^*) < 0$, then for any t^* where RZF vanishes, the corresponding zero is off the critical line, hence all nontrivial zeros are necessarily located off the critical line. This is not true, since a large number of zeros have proved to be on the critical line starting with Riemann's first few numerically computed zeros. Hence the only valid necessary optimality condition is $g(\sigma^*) = 0$, that is $\sigma^* = ½$ for any t^* where RZF vanishes. Conditions (4) and (6) are then satisfied. The necessary optimality conditions are all satisfied, for $\mu^* = 0$, and $\sigma^* = ½$. Hence a necessary condition for optimality, that is the necessary condition for RZF to vanish at any t^*, is that the corresponding nontrivial zero is on the critical line. This results proves the Riemann Hypothesis true.

A noteworthy observation is that in the proposed approach to identifing the location of RZF's nontrivial zeros, the analysis did not require the use of a closed form expression of RZF and its derivatives, but used instead a set of RZF's properties which were sufficient to show that RZF's nontrivial zeros are necessarily located on the critical line. Hence, the same result is valid for any complex-valued function that has the same properties as RZF[10]. As an example, the Riemann $\xi(s)$ function also has its zeros located on the critical line [5].

Conclusion

In this paper, the search for the location of the nontrivial zeros of the Riemann Zeta function is implemented using an optimization approach, which also served as the basis for proving several scientific laws and theories. The properties of RZF and those of its nontrivial zeros enabled the formulation of the search for their location as a constrained optimization problem using the simple objective function of minimizing the saquared norm of RZF at some heitght $t = t^*$ where it vanishes, under the constraint that nontrivial zeros are located to the left half of the critical line. Zeros on the right side of the critical line are obtained by symmetry about this line. The Kuhn-Tucker necessary optimality conditions of the resulting constrained nonlinear programming problem proved that any nontrivial zero of RZF is necessarily located on the critical line, thus proving the conjecture stated in the Riemann Hypothesis. This result is also valid for functions that have the same properties as RZF.

Appendix:

Some relevant Properties of RZF and its nontrivial zeros

The most important and relevant properties of RZF [6] are listed below:

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[10] *See the appendix for a list of RZF's pertinent properties*

1. Since RZF is analytic in the complex plane except for a pole at $\sigma=1$, its real and its imaginary parts, *U(s)* and *V(s)* respectively, are differentiable in the critical strip hence:
   $$U^*U^*_\sigma + V^*V^*_\sigma = 0$$
2. RZF has an infinite number of nontrivial zeros
3. A huge number of nontrivial zeros proved to be located on the critical line
4. As a consequence of the functional equation, nontrivial zeros either occur on the critical line or in pairs off of the critical line symmetrically about it.
5. Nontrivial zeros are located in the critical strip at different heights $t = t^*$
6. Nontrivial zeros are symmetric about the real line $t = 0$, and about the critical line
7. As per (5), if σ^* is a location of a nontrivial zero at $t = t^*$, then $(1- \sigma^*)$ is also a location of a nontrivial zero at $t = t^*$.
8. RZF has no zeros on the line $\sigma = 1$. Thus, by symmetry about the critical line, RZF has no zero on the line $\sigma = 0$, hence for nontrivial zeros: $\sigma > 0$

Properties (3) and (6) enable limiting the search for the location of RZF's nontrivial zeros to the left half of the critical strip since zeros on the right of the critical line can be derived by symmetry about this line. This leads to the following constraint: $\sigma \leq ½$

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