Using Cantor's Diagonal Method to Show $\zeta(2)$ is Irrational

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Abstract

We look at some of the details of Cantor's Diagonal Method and argue that the swap function given does not have to exclude 9 and 0, base 10. We also puzzle out why the convergence of the constructed number, its value, is of no concern. We next review general properties of decimals and prove the existence of an irrational number with a modified version of Cantor's diagonal method. Finally, we show, with yet another modification of the method, that $\zeta(2)$ is irrational.

Introduction

Cantor's diagonal method is typically used to show the real numbers are uncountable [2, 3]. Here is the reasoning.

If the reals are countable they can be listed. In particular the decimal, base 10 versions of the real numbers in the open interval $(0, 1)$ can be listed. List these numbers. Then starting with the upper left hand corner digit, construct, going down the upper left to lower right diagonal, a decimal not in the list. Use the following guide: if the decimal is 7 make your decimal 5 and if it is anything other than 7 make it 7. The number you construct is not in the list. This follows as the number constructed, per the construction, differs from every number in the list at least at one decimal place. The only exception to the uniqueness of these decimal representations occurs with rational numbers: $.2\overline{0} = .1\overline{9}$, but because our swap function doesn't generate any 0s or 9s in the constructed number we are assured our constructed number is not in the list. Therefore the real numbers in $(0, 1)$ are uncountable and a fortiori $\mathbb{R}$ is uncountable.

Could use 0 and 9

It is not difficult to see why even with a swap function involving 0 and 9, the construction still works. Every list will have a repetition of all combinations after any finite number in the list. More formally, consider the function $f(n,m)$ that gives the decimal digit of the nth number in the list at the mth position. The diagonal of any list is given by $f(n,n)$. This function does not turn into a constant, say 9, as this would indicate there are no more numbers with a not 9 value along the diagonal, but there are such numbers. We can construct such valid decimals at will in at least a countable number of ways. For example starting at (n,n) make 91, 92, 93, etc. That is just start counting $1, 2, 3, \ldots$. All will be valid decimal numbers. As there are an infinite number, it is impossible that such numbers stop at some point in the list.

By making the swap with numbers like 5 and 4 or 3 and 7 or any two that are not 9 and 0, we don't have to reason this out.

What about convergence?

Cantor's diagonal method does not address the convergence of the decimal representation of a real number constructed. Could it be all 5's ($.\overline{5}$) and hence converging to a rational number – a number in the list. A combination of 5's and 7's that represent an infinitely repeating decimal? These observations are of no concern because the argument is that the number's representation is not in the list. Statements beyond this seem irrelevant.

Of course if we suppose that ambiguity of representation is not allowed: only finite decimal representations are given of numbers like $.5\overline{0}$, the finite version, and $.4\overline{9}$, the infinite version, then the infinite decimal we construct might be an excluded infinite decimal version of a number included in the list. This is when the use of not 9 and not 0 fixes the situation fast. One could do a reductio ad absurdum argument. Suppose the constructed number converges to a number in the list, but the number in the list differs by at least one decimal point. So how close can $.5554445454\ldots$ get to say $.555444454\ldots$ – they differ at the 7th place. The numbers must differ by at least .0000001. Another argument: decimal representations are unique, excluding representations like $.5\overline{0} = .4\overline{9}$, but such a situation is impossible when neither 9 nor 0 are used in the swap function – there are no 0s or 9s in the constructed number.

But, all of these convulsive reasonings are superfluous: we can have redundancy in the representation of the numbers. Both $.5\overline{0}$ and $.4\overline{9}$ can be included in the list: in fact, the list is succinctly and efficiently given by all combinations of $.x_1x_2\ldots$ with $x_k \in \{0, 1, \ldots, 9\}$. Then any pair, indeed 0 and 9, will do per the above reasoning. You can never run out of decimals with their nth decimal of a certain value. In all cases the program works: a list of finites only; a list of finites terminating in all 9's; and a mixture of both types.

Constructing an irrational number

Curiously, Cantor, arguably, is most famous for his diagonal method and his construction of a transcendental number. The two are connected. He proved that all algebraic numbers are countable. If one lists all algebraic numbers then uses Cantor's diagonal method (henceforth CDM), we see that numbers exist that are not algebraic (not in the list): the number is a transcendental number [5]. It is rather curious that one is at once constructing a transcendental number, but ending up with just a number only in theory. It is difficult to list all algebraic numbers in a systematic way [4]. This is to be contrasted with Liouville's for real construction of a transcendental number years before Cantor's proof that they must exist [5], in spirit, so to speak. Hardy gives the history succinctly: first just one by arduous construction; then an infinity of them with Cantor's diagonal method; and then specific interesting instances with Hermite and Lindemann's proofs that e and π are transcendental [5].

It is also curious that no one, apparently till now, has thought to use CDM to prove the existence of an irrational number. This is most likely because the existence of irrational numbers was never in contention. They are a type of algebraic number and proofs that specific numbers like $\sqrt{2}$ are irrational are relatively easy. There would seem to be little point in proving the existence of irrational numbers using CDM or any other means. All of this said, here's the idea.

List all the rational numbers in $(0, 1)$ using base 10, or any other decimal base. Hardy gives a nice treatment of decimal bases in his Chapter 9 [5]. The list will include pure repeating decimals, finite decimals, and mixed decimals. In base 10, 1/3, 1/4, and 1/6 are examples of each respectively. Irrational numbers are non-repeating infinite decimals. Use the swap function that swaps or writes 5 if the number encountered using CDM is not 5 and 7 if the number encountered is 5. The number constructed is not in the list; it differs by at least one decimal point from all numbers listed. As all the numbers are all the rationals in $(0, 1)$ and the number generated is in $(0, 1)$ it must be irrational. The number will be a non-repeating infinite decimal consisting of a string of 5s and 7s, an irrational.

The same analysis applied to show that with Cantor's original argument the ambiguous decimals $(.4\overline{9} = .5\overline{0})$ and the convergence of the number constructed are not relevant apply to this proof as well. There are countably infinite decimals of the form $f(n, n) = x$, the diagonal decimals, where $x \in \{0, 1, 2, \ldots, 9\}$. So any construction that has a finite number of 0 and then all 9's, if we used 0 and 9 in our swap function, is impossible.

Using addition in CDM

The swap function seems a little arbitrary in nature. We will show that it can be replaced by additions with the good effect that $\zeta(2)$ and other numbers can be proven to be irrational.

As a warm-up to proving $\zeta(2)$ is irrational, we will prove that all rational numbers can't be written as a finite decimal in base 4. We will use a modified version of CDM.

In Table 1, we have a list of all single decimals in base 4^k in $(0, 1)$: that is

$$D_{4^k} = \{1/4^k, 2/4^k, 3/4^k, \ldots, (4^k - 1)/4^k\}$$

in Table 1. Each D_{4^k} will include new numeric values as well as all values in previous D_{4^m}, where $m < k$. So given a finite decimal of length r in base 4 it will be an element of D_{4^r} and hence in the list. Numeric values are repeated infinitely often. For example $1/4 \in D_{4^r}$ for all $r \geq 1$.

Now using addition, instead of Cantor's swap function, we construct a decimal, $.\overline{1}$. Table 2 shows the procedure. Each column's partial sum is excluded not only from the column's decimal set, but all the previous decimal sets to the left of the present column. For example, $1/4 + 1/4^2 \notin D_4$ and $1/4 + 1/4^2 + 1/4^3 \notin D_{4^2}$ and also this partial is not in D_4 as well.

Each total requires, per scientific notation, greater and greater precision in a fixed base of the form 4^n or a greater power of 4. We

D_4							
	D_{4^2}						
		D_{4^3}					
			D_{4^4}				
				D_{4^5}			
					$\ddots$		
						$D_{4^{(k-1)^2}}$	
							$\ddots$

Table 1: A list of all finite decimals base 4.

$+1/4$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$+1/4^2$	$+1/4^2$	$+1/4^2$	$+1/4^2$	$+1/4^2$	$\ldots$	$+1/4^2$	
$\notin D_4$	$+1/4^3$	$+1/4^3$	$+1/4^3$	$+1/4^3$	$\ldots$	$+1/4^3$	
	$\notin D_{4^2}$	$+1/4^4$	$+1/4^4$	$+1/4^4$		$\vdots$	
		$\notin D_{4^3}$	$+1/4^5$	$+1/4^5$		$\vdots$	
			$\notin D_{4^4}$	$+1/4^6$		$\vdots$	
				$\notin D_{4^5}$			
						$+1/4^{(k-1)^2}$	
						$+1/4^{k^2}$	
						$\notin D_{4^{(k-1)^2}}$	
							$\ddots$

Table 2: A list of all finite decimals base 4. The decimal number $.\overline{1}$, base 4 is generated by the sums.

observe the first few totals are given by 1.0×4^{-1}; 1.1×4^{-1}; 1.11×4^{-1}; 1.111×4^{-1}. That is uses a fixed base, base 4, each total requires more precision. Increasing the power of 4, say to a base of $4^2 = 16$, we express the total of $1/4 + 1/16$ with 5.0×16^{-1}. But any such power will require more precision for subsequent totals. As this is an infinite sum, a infinite series, the need for greater precision never ends. No finite decimal can accommodate infinite precision, an infinite decimal: $.\overline{1}$.

Collaborating this: we know $.\overline{1}$, base 4 is the infinite geometric series

$$\sum_{k=1}^{\infty} \frac{1}{4^k},$$

which converges to 1/3. Hardy gives a proof that in general a fraction with a denominator of d will require an infinite repeating decimal in base r if $(r, d) = 1$, that is, the denominator of the fraction and the base are relatively prime [5]; we observe $(3, 4) = 1$.

From a different set topology angle: let C_x be the best approximation to 1/3 in D_{4^x}, where x is a natural number: $|C_x - 1/3| \neq 0$. As the set of partials are the best approximations to 1/3 in these decimal sets, the partials taken collectively are an infinite set and 1/3 is a limit point – it must be outside all decimal sets.

Here's yet another set theory angle. In what follows $\mathbb{R}(0, 1)$ are the real numbers in $(0, 1)$. We have

$$\sum_{k=1}^{n} \frac{1}{4^n} \notin \bigcup_{k=1}^{n-1} D_{4^k},$$

(this is Table 2). We can infer

$$\sum_{k=1}^{n} \frac{1}{4^n} \in \mathbb{R}(0,1) \setminus \bigcup_{k=1}^{n-1} D_{4^k},$$

and, taking the limit as $n \to \infty$, this gives

$$\sum_{k=1}^{\infty} \frac{1}{4^n} \in \mathbb{R}(0,1) \setminus \bigcup_{k=1}^{\infty} D_{4^k} = \mathbb{R}(0,1) \setminus \mathbb{F}(0,1),$$

where $\mathbb{F}(0, 1)$ are finite decimals in base 4. Note: $1/3 \in \mathbb{R}(0, 1)\backslash\mathbb{F}(0, 1)$. The rational number 1/3, or $.\overline{1}$ can't be written as a finite decimal in base 4.

Proving $\zeta(2)$ is irrational

The irrationality of $\zeta(2)$ and indeed $\zeta(2n)$ has long been established. Both follow from the identity

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{p-1} \frac{2^{2n-1}}{(2n!)} B_{2n} \pi^{2n} \tag{1}$$

where B_{2n} are the Bernoulli numbers. As Bernoulli numbers are rational and π is transcendental, this identity shows $\zeta(2n)$ is irrational, n a natural number. A proof of the irrationality of $\zeta(3)$ was given by Apery [1]. Apart from this result, there are less than satisfactory results concerning other odd zeta values: there are infinitely many odd irrationals [7] and one or more of $\zeta(3, 5, 7, 9, 11)$ is irrational [8].

D_4						
	D_9					
		D_{16}				
			D_{25}			
				$\ddots$		
					D_{k^2}	
						$\ddots$

Table 3: $\mathbb{Q}(0, 1)$

In Table 3 is a modified Cantor's Diagonal Table. The symbols D_{n^2} give single decimal points in base n^2. So, for example $D_4 = \{.1, .2, .3\}$ in base 4. All rational numbers in $(0, 1)$, $\mathbb{Q}(0, 1)$, are represented in the union of these sets. This follows as for any rational $0 < p/q < 1$, $p/q = pq/q^2$ with $p < q$: that is $p/q \in D_{q^2}$.

We will take as a given the set exclusions in Table 4. They are certainly plausible. For proofs see [6].

As with Tables 1 and 2, the set exclusions are cumulative in Table 4. For example, $1/4+1/9 \notin D_4$ and $1/4+1/9+1/16$ is not in D_4, D_9, or D_{16}. So, like Cantor's diagonal method as applied to a list of base ten decimals, we build, not with a swap function, but with partial sums, a number not in any decimal base given by a single decimal base n^2. It is clear, that all bases, like base 4 are given by the union of sets, for base 4, the union of D_{4^k} sets.

We must have that $z_2 = \zeta(2) - 1$ requires an infinite decimal in all bases; it's not in the list of all rationals in $(0, 1)$ and $z_2 \in (0, 1)$. As argued, the ambiguity of decimals and the convergence of the constructed number – our sum z_2 – are both immaterial to the method. We notice that z_2 is a limit point of

$$\bigcup_{k=2}^{\infty} D_{k^2}$$

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\dots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\dots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 4: All rationals are excluded via partial sums of $\zeta(2) - 1$.

and hence is not in this set of all rationals. This parallels our treatment of $.\overline{1}$ base 4 in the previous section.

The argument can be succinctly stated using set theory: given

$$\sum_{k=2}^{n} \frac{1}{k^2} \notin \bigcup_{k=2}^{n} D_{k^2},$$

that is given Table 4, we can infer

$$\sum_{k=2}^{n} \frac{1}{k^2} \in \mathbb{R}(0,1) \setminus \bigcup_{k=2}^{n} D_{k^2},$$

and, taking the limit as $n \to \infty$, this gives

$$\sum_{k=2}^{\infty} \frac{1}{k^2} \in \mathbb{R}(0,1) \setminus \bigcup_{k=2}^{\infty} D_{k^2} = \mathbb{H}(0,1),$$

where $\mathbb{H}(0,1)$ are the irrational numbers in the interval $(0,1)$ and $\mathbb{R}(0,1)$ are the reals in $(0,1)$. This implies z_2 is irrational.

It is a question of precision: $\zeta(2) - 1$ requires infinite precision. The screens of the decimals get so fine any rational number would be caught or blocked, but $\zeta(2) - 1$ is not caught or blocked with these rational screens – by our sum construction.

Relationship between the two

We have developed CDM, based on a single decimal base (one with a swap function and one with a sum), and a modified version of CDM, call it CDMM, based on effectively all decimal bases and using a sum. One could pose that the relationship between the proof that there is a number that can't be given by a finite decimal in base 4 and the $\zeta(2)-1$ development with this: there is a number that can't be written as a finite decimal in any base. Per Hardy's proof that an infinite repeating decimal is required in a base depending on the denominator and the base, they being relatively prime, such a number must be irrational. There is no rational number's denominator that is relatively prime to all bases, all $n > 1$, natural numbers. Note: n^2 (and n^p, $p > 2$ a natural number) has the same primes as n. If one could forget for a moment the idea of proving $\zeta(n)$ is irrational, n a natural number greater than 1 and focus on constructing a number that can't be given as a finite decimal in any number base, one could see the logic of the argument given. It is just a generalization of CDM as applied to proving the existence of a value requiring an infinite decimal in base 4 given above. We repeat the argument with all decimal bases n^2. As one crosses, so to speak, 4^r in Table 4, it is clear that the number being constructed will require more than r decimal places; the number is not in any of the sets D_m, $m \leq r$. The build always escapes the finer and finer net of $1/n^2$ decimals. The number the partials converges to, by our sum construction, requires infinite precision in all n^2, hence n bases: an attribute of an irrational number.

Conclusion

In this article we have stretched Cantor's diagonal method in a new direction. Our result generates new decimals, not in given list, not by changing a decimal digit down a diagonal, but by changing, really, potentially lots of decimal digits in various places. It thus generalizes the notion of you must use the diagonal to you can use any change anywhere as long as you maintain that the new number is not equal to the current listed number or any numbers encountered previously in the list. It should be remembered that infinite decimals are infinite series and z_2's terms are all single decimals in n^2 bases. We could just translate Table 3 to base 10 decimals.

That said, this new idea does not threaten Cantor's original use of his diagonal method. We can't prove with our use of sums generating guaranteed changes that the real numbers are uncountable. We need an unambiguous list (the rationals); the type we get with decimal sets. The summation replacement of the swap function has limited applicability, but, it does give another way to show the irrationality of certain infinite series. All $\zeta(n)$, n an even or odd number greater than 1, gives an instance. Tables 3 and 4 are valid for such $\zeta(n)$; the prime factorizations are the same for n and any natural number power of n. There are many such infinite series, lots with known convergence points generally involving π and e, known irrational numbers. We now have for every such series, potentially, a new irrationality proof. Using (1), we thus have a proof that all even (with a little work odd too) powers of π are irrational.

References

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