Using Cantor's Diagonal Method to Show $\zeta(2)$ is Irrational

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Abstract

We look at some of the details of Cantor's Diagonal Method and argue that the swap function given does not have to exclude 9 and 0, base 10. We next review general properties of decimals and prove the existence of an irrational number with a modified version of Cantor's diagonal method. Finally, we show, with yet another modification of the argument, that $\zeta(2)$ is irrational.

Introduction

Cantor's diagonal method is typically used to show the real numbers are uncountable [1, 2]. Here is the reasoning.

If the reals are countable they can be listed. In particular the decimal, base 10 versions of the real numbers in the open interval $(0, 1)$ can be listed. List these numbers. Then starting with the upper left hand corner digit, construct, going down the upper left to lower right diagonal, a decimal not in the list. Use the following guide: if the decimal is 7 make your decimal 5 and if it is anything other than 7 make it 5. The number you construct is not in the list. This follows as the number constructed, per the construction, differs from every number in the list at least at one decimal place. The only exception to the uniqueness of these decimal representations occurs with rational numbers: $.2\overline{0} = .1\overline{9}$, but because our swap function doesn't generate any 0s or 9s in the constructed number we are assured our constructed number is not in the list. Therefore the real numbers in $(0, 1)$ are uncountable and a fortiori $\mathbb{R}$ is uncountable.

Could use 0 and 9

It is not difficult to see why even with a swap function involving 0 and 9, the construction still works. One must contrive a list of real numbers in $(0, 1)$ in a particularly pernicious order. Every *nth* position after a point must be not 9 in order to build a string of all 9's. If this could be true of a list, a number like $.00000\overline{9}$ could be the one constructed. If we observe .00001 in the list we have not constructed a number not it the list. But for any *nth* position there must be an infinite number with all possible digits, $\{0, 1, \ldots, 9\}$, at that position. Hence, after working down the list, to say the *mth* number, there is a number further down that will block, in effect, any construction from being repeated. Every list will have a repetition of all combinations after any finite number in the list.

By making the swap with numbers like 5 and 4 or 3 and 7 or any two that are not 9 and 0, we don't have to reason this out.

What about convergence?

Cantor's diagonal method does not address the convergence of the decimal representation of a real number constructed. Could it be all 5's ($.\overline{5}$) and hence converging to a rational number – a number in the list. A combination of 4's and 5's that represent a infinitely repeating decimal? These observations are of no concern because the argument is that the number's representation is not in the list. Statements beyond this seem irrelevant.

Of course if we suppose that ambiguity of representation is not allowed: only finite decimal representations are given of numbers like .5 and $.4\overline{9}$, then the infinite decimal we construct might be an excluded infinite decimal version of a number included in the list. This is when the use of not 9 and not 0 fix the situation fast. One could do a reductio ad absurdum argument. Suppose the constructed number converges to a number in the list, but the number in the list differs by at least one decimal point. So how close can $.5554445454\ldots$ get to say $.555444454\ldots$ – they differ at the 7th place. The numbers must differ by at least .0000001. Another argument: decimal representations are unique, excluding representations like $.5\overline{9} = .6$, but such a situation is impossible when neither 9 nor 0 are used in the swap function – there are no 0s or 9s in the constructed number.

But, all of these convulsive reasonings are superfluous: we can have redundancy in the representation of the numbers. Both $.5\overline{0}$ and $.4\overline{9}$ can be included in the list: in fact, the list is succinctly and efficient given by all combinations of $.x_1x_2\ldots$ with $x_k \in \{0, 1, \ldots, 9\}$.

Constructing an irrational number

Curiously, Cantor is most famous for his diagonal method and his construction of a transcendental number. The two are connected. He proved that all algebraic numbers are countable. If one lists all algebraic numbers then uses Cantor's diagonal method (henceforth CDM) shows there exists numbers that are not algebraic (not in the list): the number is a transcendental numbers [3]. It is rather curious that one is at once constructing a transcendental number, but ending up with just a number only in theory. You can't list all algebraic numbers. This is to be contrasted with Liouville's for real construction of a transcendental number years after Cantor's original proof that they must exist [3], in spirit, so to speak.

It is also curious that no one, apparently till now has thought to use CDM to prove the existence of an irrational number. This is most likely because irrational numbers are a type of algebraic number and proofs that specific numbers like $\sqrt{2}$ are irrational are relatively easy. There would seem to be little point in proving the existence of irrational numbers using CDM or any other means. All of this said, here's the idea.

List all the rational numbers in $(0, 1)$ using base 10, or any other decimal base. Hardy gives a nice treatment of decimal bases in his Chapter 9 [3]. The list will include pure repeating decimals, finite decimals, and mixed decimals. In base 10, 1/3, 1/4, and 1/6 are examples of each respectively. Use the swap function that swaps or writes 4 if the number encountered using CDM is not 4 and 5 if the number encountered is 4. The number constructed is not in the list; it differs by at least one decimal point from all numbers listed. As all the numbers are all the rationals and the number generated is in $(0, 1)$ it must be irrational. The number will be a non-repeating infinite decimal, an irrational.

Proving $\zeta(2)$ is irrational

In Table 1 is a modified Cantor's Diagonal Table. The symbols D_{n^2} give single decimal points in base n^2. So, for example $D_4 = \{.1, .2, .3\}$ in base 4. How to read the table: All previous columns (left to right) pertain to the new, right most partial. For example $1/4 + 1/9 + 1/16$ is not in D_4, D_9, or D_{16}. So, like Cantor's diagonal method as applied to a list of base ten decimals, we build, not with a swap function, but with an addition, a number not in any decimal base given by a single decimal base n^2.

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\ldots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 modified to exclude them all via partial sums of $\zeta(2) - 1$.

If this is true, can we conclude that $z_2 = \zeta(2) - 1$ must be irrational? Note: for any rational $0 < p/q < 1$, $(pq)/q^2 \in D_{q^2}$. Thus all partials with a last term of $1/q^2$ can't be given with any single decimal base m^2 with $m < q$. Are we building a infinite series, like the infinite decimal of Cantor's original argument, that must be excluded from the list of all rationals (here-to-for all reals) and thus be irrational? Does the elimination element of Cantor's Diagonal Method force an irrational sum? Like CDM (in its original use) can we ignore the convergence point of the built infinite series?

Well, to play it safe, can we prove the convergence point is not in our list? Consider the following use of the triangle inequality: let C_x

be a single decimal rational in some D_{m^2}, the best, meaning closest to z_2 in D_{m^2}, then for all n (and m !!!) large enough,

$$0 \leq \left| C_x - \sum_{k=2}^{n} \frac{1}{k^2} \right| < \epsilon/2. \tag{1}$$

and

$$0 < \left| \sum_{k=2}^{n} \frac{1}{k^2} - z_2 \right| < \epsilon/2.$$

So, in all cases,

$$0 < |C_x - z_2| < \epsilon.$$

But this says z_2 is not rational. Note: any given rational number is repeated infinitely many times in D_{k^n}. For example, all rationals with denominators less than n are contained in $D_{(n!)^2}$. The best approximation of z_2 in any D_{m^2} is never exact and later higher powers of m^2 are better.

The geometric series

In Table 2, the same technique as given for z_2 is used to show $\overline{1}$ base 4 is not a finite decimal base 4. We know this, apart from Table 2, because the geometric series associated with this infinite, repeating decimal is

$$\sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3}.$$

$+1/4$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$+1/4^2$	$+1/4^2$	$+1/4^2$	$+1/4^2$	$+1/4^2$	$\ldots$	$+1/4^2$	
$\notin D_4$	$+1/4^3$	$+1/4^3$	$+1/4^3$	$+1/4^3$	$\ldots$	$+1/4^3$	
	$\notin D_{4^2}$	$+1/4^4$	$+1/4^4$	$+1/4^4$		$\vdots$	
		$\notin D_{4^3}$	$+1/4^5$	$+1/4^5$		$\vdots$	
			$\notin D_{4^4}$	$+1/4^6$		$\vdots$	
				$\notin D_{4^5}$			
						$+1/4^{(k-1)^2}$	
						$+1/4^{k^2}$	
						$\notin D_{4^{(k-1)^2}}$	
							$\ddots$

Table 2: A list of all finite decimals base 4. The decimal number $.\overline{1}$, base 4 is generated by the sums.

Looking again at Table 1, one can see why the technique shows that z_2 is irrational. In order for the partials to be converging to a rational, for every ϵ, there would have to be a rational that all partials with a given upper limit greater than n gets close to. But the partials always exceed, with their denominator, all $\{2, 4, \ldots, n^2\}$ denominators. There are denominators further out from any given, fixed denominator that get consistently closer to such partials. Limits are unique, so the limit point can't be both approaching a previous finite decimal excluded earlier and some other later decimal.

Another way to consider the situation is to note that approximations to limit points can never be limit points themselves. Finite decimals base 4 are all approximations to 1/3, some better, some worse, none equal. In the case of $\zeta(2) - 1$ all rationals are approximations, some better, some worse, none perfect – that is equal to $\zeta(2) - 1$. This means that $\zeta(2)$ is irrational.

Yet from another perspective, each increase in the power of n^2, from n^2 to n^4 to n^6, in $D_{n^{2k}}$ yields a better approximation that supersedes the previous power. That is you really need to use the refinement of the ruler's tick marks to get a better approximation than the previous power of n yielded. This means that an infinite decimal is needed in all bases, all n^2s, but, as the prime divisor of n and n^2 are the same, this implies that an infinite decimal is needed for z_2 in all bases. This is only possible if z_2 is irrational.

Attempting the reasoning used in (1), we have $1/3 \in D_3$ and it is fixed, unlike $\zeta(2)$ that requires differing C_x rationals. For sufficient n, $1/3$ will give a better approximation to the partial than any finite decimal in base 4. With $\zeta(2)$ there is no equivalent of $1/3$ that is being approached.

Evens and Bertrand's postulate

It remains to show that partials escape the denominators of their terms: the set exclusions in the columns of Table 1 are correct. This is the juncture of the argument where the fact that all numbers (greater than 1, to a power of 2) are included in the sum of z_2. Every other number is even so the power of 2 in the denominator is always greater than 2 and so the reduced denominator always has at least 2 in it. For example,

$$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{61}{144}$$

and

$$\frac{61}{144} + \frac{1}{25} = \frac{25 \times 61 + 1 \times 144}{144 \times 25}$$

shows how the pattern continues.

Using Bertrand's postulate [3], we know there exists a prime p between $n^2/2$ and n^2.

Putting these two results together,

$$\sum_{k=2}^{n} \frac{1}{k^2} = \frac{a}{b},$$

where a/b is a reduced fraction with $b > n^2$. This established the set exclusions in the columns of Table 1 are correct. For details and general $\zeta(n)$, $n \geq 2$, see [4].

Conclusion

Cantor's diagonal method applied to show the existence of an irrational number and the proof given here for the irrationality of $\zeta(2)$ can be viewed as the same. The negations of set inclusions in Table 1 show that somewhere the decimal associated with the partial is not the same as those in each D_{k^2} set. As the union of all such sets give all the rationals the irrationality of $\zeta(2)$ follows. All decimal bases n^2 are replicated, including base 10 – just like Cantor's original idea. You don't have to use a diagonal to ensure there is no replication: the wash just requires that as you go down the list, the construction can't equal any previously encountered number.

References

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[2] R. Courant, H. Robbins, *What is Mathematics*, Oxford University Press, London, 1948.

[3] G. H. Hardy, E. M. Wright, R. Heath-Brown, J. Silverman, and A. Wiles, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, London, 2008.

[4] T.W. Jones, A Simple Proof that $\zeta(n)$ is Irrational (2017), available at `available at http://vixra.org/abs/1801.0140`