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Department of Mathematics

A Termpapaper on:

RIEMANN INTEGRATION ON $\mathbb{R}^n$

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In

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A project work submitted by

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1 Riemann Integration

Throughout these discussions the numbers $\epsilon > 0$ and $\delta > 0$ should be thought of as very small numbers. The aim of this part is to provide a working definition for the integral $\int_a^b f(x)dx$ of a bounded function $f(x)$ on the interval [a, b]. We will see that the real number $\int_a^b f(x)dx$ is really the limit of sums of areas of rectangles.

1.1 Partitions and Riemann sums

1.1.1 Definition (Partition P_δ of size $\delta > 0$)

Given an interval $[a, b] \subset \mathbb{R}$, a partition P_δ denotes any finite ordered subset having the form

$$P_\delta = [a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b]$$

where

$$\delta = \max[x_i - x_{i-1} \mid i = 1, \cdots, n]$$

denotes the distance between any two adjacent partition point x_{i-1} and x_i, and where n denotes the number of subintervals that [a, b] is partitioned into, with n depending on δ so that $n = n(\delta)$. The simplest partitions have uniform spacing between partition points, in which case $\delta = \frac{b-a}{n}$ or conversely, $n(\delta) = \frac{b-a}{\delta}$

1.1.2 Definition (Selection of evaluations points z_i)

The evaluations points z_i are a collection of n points in the interval [a, b] such that

$$[x_0 \leq z_1 \leq x_1 \leq z_2 \leq x_2 \leq \cdots \leq x_{n-1} \leq z_n \leq x_n]$$

Having a partition P_δ of the interval [a, b] and having chosen the set of n evaluation points $z_1, z_2, \cdots, z_n$, we can now define the so-called Riemann sum.

1.1.3 Definition (Riemann sum for the function $f(x)$)

Given a function $f : [a, b] \to \mathbb{R}$, a partition P_δ, and a selection of evaluation points z_i, the Riemann sum of f is denoted by

$$S_\delta(f) = \sum_{i=1}^{N} f(z_i)(x_i - x_{i-1})$$

Again, note that since the choice of the evaluation points z_i is arbitrary, there are infinitely many Riemann sums associated with a single function and a partition P_δ.

1.1.4 Definition (Integrability of the function $f(x)$)

The function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if for all $\epsilon > 0$, we can choose $\delta > 0$ sufficiently small so that

$$|S_\delta(f) - S(f)| < \epsilon$$

for any Riemann sum $S_\delta(f)$ with maximum partition width δ.
Whenever the limit $S(f)$ exists we say that $S(f)$ is the integral of $f(x)$ over the interval [a, b] and write

$$\int_a^b f(x)dx = S(f) = \lim_{\delta \to 0} S_\delta(f)$$

Thus, $\int_a^b f(x)dx$ is just a limit of Riemann sums $S_\delta(f)$ whenever such a limit exists.

1.1.5 Definition (Notation for integrable functions)

We let

$$R(a,b) = [f : [a,b] \to \mathbb{R} \mid f \text{ is Riemann integrable}]$$

1.2 Upper and Lower Riemann Sums

The integrability requirement given above is too general, and we can redefine it so as to make it more practical. This leads us to the concept of the lower and upper Riemann sum, known also as the lower and upper Darboux sum. The notion is to fix the selection points z_i so as to select two particular Riemann sums.

1.2.1 Definition (M_i and m_i)

Given a partition P_δ, for $i = 1, \cdots, N(\delta)$, we set

$$M_i = \sup f(x)$$

$$x \in [x_{i-1}, x_i]$$

and

$$m_i = \inf f(x)$$

$$x \in [x_{i-1}, x_i]$$

1.2.2 Definition (Upper and Lower Riemann Sums)

Given a partition P_δ, we let

$$U_\delta(f) = \sum_{i=1}^{N} M_i(x_i - x_{i-1})$$

and

$$L_\delta(f) = \sum_{i=1}^{N} m_i(x_i - x_{i-1})$$

denote the upper and lower Riemann sums respectively.

1.2.3 Definition (Integrability of $f(x)$ in terms of L(f) and U(f))

$f \in$ R(a, b) if L(f) = U(f). In this case,

$$\int_a^b f(x)dx = L(f) = U(f)$$

While this definition may be different from our former definition, on the other hand, this definition is much easier to compute with.

1.2.4 Example (Compute $\int_0^1 xdx$)

We subdivide [0, 1] into n subintervals, with partition width $\delta = \frac{1}{n}$. It follows that for $i = 1, 2, \cdots, n,\ m_i = x_{i-1} = \frac{i-1}{n}$ *and* $M_i = x_i = \frac{i}{n}$. Then,

$$L_\delta(f) = \sum_{i=1}^{n} \frac{i-1}{n}\frac{1}{n} = \frac{(n-1)n}{2n^2}$$

and

$$U_\delta(f) = \sum_{i=1}^{n} \frac{i}{n}\frac{1}{n} = \frac{(n+1)n}{2n^2}$$

Since $L(f) = \lim\limits_{\delta\to 0} L_\delta(f) = \frac{1}{2}$ and $U(f) = \lim\limits_{\delta\to 0} U_\delta(f) = \frac{1}{2}$, we see that $\int_0^1 xdx = L(f) = U(f) = \frac{1}{2}$

1.2.5 Definition (Refinement of Partitions)

If $P_\delta \subset Q_\delta$, then Q_δ is a refinement of P_δ

1.3 Properties of Upper and Lower Sums

1. If $f : [a, b] \to \mathbb{R}$ is bounded and P_δ and Q_δ are two partitions of [a,b] such that $P_\delta \subset Q_\delta$, then
$$L_\delta(P) \leq L_\delta(Q) \leq U_\delta(Q) \leq U_\delta(P)$$

2. If $f : [a, b] \to \mathbb{R}$ is bounded, then for any two partitions P_δ and Q_δ,
$$L_\delta(P) \leq U_\delta(Q)$$

3. If $f : [a, b] \to \mathbb{R}$ is bounded, then
$$L(f) \leq U(f)$$

1.3.1 Theorem (Cauchy Criterion for Integrability in Terms of Upper and Lower Sums)

A bounded function $f \in R(a, b)$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$U_\delta(f) - L_\delta(f) < \epsilon$$

1.4 The Riemann Integral is Linear

Since the Riemann integral is defined as the infinite limit of a sequence of finite sums, and as summation is linear operation, we expect that limiting integral should also be linear.

Suppose $f, g \in R(a, b)$. Then

1. $f + g \in R(a, b)$ and $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

2. For all $c \in R$, $cf \in R(a, b)$ and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

1.5 Further Properties of the Riemann Integral

1. If $f, g \in R(a,b)$ and $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\int_b^a f(x)dx \leq \int_a^b g(x)dx$$

2. Suppose $f : [a,b] \to \mathbb{R}$. Let c be any point in (a,b). If $f \in R(a,b)$ and $f \in R(c,b)$, then $f \in R(a,b)$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

1.5.1 Theorem (Fundamental Theorem of Calculus)

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then f has an anti-derivative, F and

$$F(b) - F(a) = \int_a^b f(x)dx$$

If G is any other anti-derivative of f, then the identity

$$G(b) - G(a) = \int_a^b f(x)dx$$

also holds.

2 Preliminaries

2.1 Definition(An interval)

A connected subset I of the topological space $\mathbb{R}$ is called an interval.

2.2 Definition

We shall say that an interval I is

(a) non-degenerate if $a < b$

(b) open if $I = (a, b)$

(c) closed if $I = [a, b]$

(d) bounded if $-\infty < a$ and $b < \infty$

2.3 Definition(Length of Interval)

The length of $I = [a, b]$ will be denoted by $|I| = b - a$ where $a < b$

2.4 Definition(δ- neighborhood of an Interval)

For an interval $I = [a, b]$ and $\delta > 0$, we shall denote by I_δ, the δ- neighborhood of I:

$$I_\delta = (a - \delta, b + \delta)$$

2.5 Cells

2.5.1 Definition(n-cell)

An n-cell is the Cartesian product of n intervals $I = I_1 \times I_2 \times I_3 \times \cdots \times I_n$.
It is naturally a subset of metric space $\mathbb{R}^n$. A cell is the same as an interval.

2.5.2 Definition

We say that an n-cell is

(a) non-degenerate if each I_j is non-degenerate.

(b) open if each I_j is open.

(c) closed if each I_j is closed.

(d) bounded if each I_j is bounded.

3 The Riemann Integral In n-Variables

We define the Riemann integral of a bounded function $f : R \longrightarrow \mathbb{R}$, where $R \subset \mathbb{R}^n$ is a cell. Recall that a partition of an interval $I = [a, b]$ is a finite collection of subintervals. The $I_1 \times I_2 \times \cdots \times I_n$ of subsets $I_1, I_2, \ldots, I_n$ of R is the set of points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ such that $x_1 \in I_1, x_2 \in I_2, \ldots, x_n \in I_n$. For example, the Cartesian product of the two closed intervals

$$[a_1, b_1] \times [a_2, b_2] = \{(x, y)|a_1 \leq x \leq b_1 \quad , \quad a_2 \leq y \leq b_2\}$$

The Cartesian product of three closed intervals

$$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] = \{(x, y, z)|a_1 \leq x \leq b_1, \quad a_2 \leq y \leq b_2, \quad a_3 \leq z \leq b_3\}$$

If $n = 1, 2, 3, \ldots, n$, then V(R) is, respectively, the length of an interval.
If $R = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ and $P_r : a_r = a_{r_0} < a_{r_1} < a_{r_2} < \cdots < a_{r_m} = b_r$ is a partition of $[a_r, b_r]$, $1 \leq r \leq n$, then the set of all rectangles in $\mathbb{R}^n$ that can be written as

$$[a_{1,j_i-1}, a_{1j_1}] \times [a_{2,j_2-1}, a_{2j_2}] \times \cdots \times [a_{n,j_n-1}, a_{nj_n}], \quad 1 \leq j_r \leq m_r, \quad 1 \leq r \leq n$$

is a partition of R. We denote this partition by $P = P_1 \times P_2 \times \cdots \times P_n$ and define its norm to be the maximum of the norms of $P_1, P_2, \ldots, P_n$,as defined in the previous section; thus,

$$||P|| = max\{||P_1||, ||P_2||, \ldots, ||P_n||\}$$

If $P = P_1 \times P_2 \times \cdots \times P_n$ and $P' = P'_1 \times P'_2 \times \cdots \times P'_n$ are partitions of the same rectangle, then P' is a refinement of P. If P'_1 is a refinement of P_i, $1 \leq i \leq n$, as defined in the previous section.
Suppose that f is a real-valued function defined in $\mathbb{R}^n$, $P = \{R_1, R_2, \ldots, R_k\}$ is a partition of R, and X_j is an arbitrary point in R_j, $1 \leq j \leq k$. Then

$$\sigma = \sum_{j=1}^{k} f(X_j)V(R_j)$$

is a Riemann sum of f over P. Since X_j can be chosen arbitrarily in R_j, there are infinitely many Riemann sums for a given function f over any partition P of R.

3.1 Definition

Let f be a real-valued function defined on a rectangle R in $\mathbb{R}^n$. We say that f is Riemann integrable if there is a number L with the following property:
For every $\epsilon > 0$, there is a $\delta > 0$ such that $|\sigma - L| < \epsilon$ if σ is any Riemann sum of f over a partition P of R such that $||P|| < \delta$. In this case, we say that L is the Riemann integral of f over R, and write

$$\int_R f(x)dX = L$$

If R is degenerate, it implies that $\int_R f(x)dX = 0$ for any function f defined on R. Therefore, it should be understood henceforth that whenever we speak of a rectangle in $\mathbb{R}^n$ we mean a non-degenerate rectangle, unless it is stated to the contrary.
The integral $\int_R f(x)dX$ is also written as

$$\int_R f(x, y)d(x, y) \quad (n = 2), \qquad \int_R f(x, y, z)d(x, y, z) \quad (n = 3)$$

or

$$\int_R f(x_1, x_2, \ldots, x_n)d(x_1, x_2, \ldots, x_n) \quad (n \quad arbitrary)$$

3.2 Upper and Lower Integrals

3.2.1 Theorem

If f is unbounded on the non-degenerate rectangle R in $\mathbb{R}^n$, then f is not integrable on R.

3.2.2 Theorems

Let f be bounded on a rectangle R and let P be a partition of R.
Then

(a) The upper sum S(P) of f over P is the supremum of the set of all Riemann sums of f over P:

(b) The lower sum s(P) of f over P is the infimum of the set of all Riemann sums of f over P

Theorem

If f is integrable on a rectangle R, then

$$\underline{\int_R} f(X)dX = \overline{\int}_R f(X)dX = \int_R f(X)dX$$

Theorem

If f is integrable on a rectangle R, then

$$\underline{\int_R} f(X)dX = \overline{\int}_R f(X)dX = L,$$

then f is integrable on R, and

$$\int_R f(X)dX = L,$$

Theorem

A bounded function f is integrable on a rectangle R if and only if

$$\underline{\int_R} f(X)dX = \overline{\int}_R f(X)dX,$$

Theorem

If f is bounded on a rectangle R, then f is integrable on R if and only if for every $\epsilon > 0$, there is a partition P of R such that

$$S(P) - s(P) < \epsilon.$$

Theorem

If f is continuous on a rectangle R in R^n, then f is integrable on R.

3.3 Properties of Riemann Integral in n Variables

(a) If f and g are integrable, then so is $f+g$, and

$$\int (f+g)(X)dX = \int f(X)dX + \int g(X)dX$$

(b) If f is integrable and c is a constant, then cf is integrable S, and

$$\int (cf)(X)dX = c\int f(X)dX$$

(c) If f and g are integrable and $f(X) \leq g(X)$ for X, then

$$\int f(X)dX \leq \int g(X)dX$$

(d) If f is integrable, then so is $|f|$, and

$$|\int f(X)dX| \leq \int |f(X)|dX$$

(e) If f and g are integrable, then so is the product fg

3.4 Iterated Integrals and Multiple Integrals

Except for very simple examples, it is tedious to evaluate multiple integrals from definitions. Fortunately, this can usually be accomplished by evaluating n successive ordinary integrals. To build the method, let us first assume that f is continuous on R=[a,b] x [c,d]. Then, for each y in [c,d], f(x,y) is continuous with respect to x on [a,b], so the integral

$$F(y) = \int_a^b f(x,y)dx$$

exists. Moreover, the uniform continuity of f on R implies that F is continuous and therefore integrable on $[c,d]$. We say that

$$I_1 = \int_c^d F(y)dy = \int_c^d \left(\int_a^b f(x,y)dx\right) dy$$

is an iterated integral of f over R. We will usually write it as

$$I_c = \int_c^d dy \int_a^b f(x,y)dx$$

Another iterated integral can be defined by writing

$$G(x) = \int_c^d f(x,y)dy \quad , \quad a \leq x \leq b$$

and defining

$$I_2 = \int_a^b G(x)dx = \int_a^b (\int_c^d f(x,y)dy)dx$$

which we usually write as

$$I_2 = \int_a^b dx \int_c^d f(x,y)dy$$

3.4.1 Example

$$Let \quad f(x,y) = x + y \quad and \quad R = [0,1] \times [1,2].$$

Then

$$f(y) = \int_0^1 f(x,y)dx = \int_0^1 (x+y)dx = (\frac{x^2}{2} + xy) \mid_{x=0}^1 = \frac{1}{2} + y$$

and

$$I_1 = \int_1^2 f(y)dy = \int_1^2 (\frac{1}{2} + y)dy = (\frac{y}{2} + \frac{y^2}{2}) \mid_1^2 = 2$$

Also,

$$G(x) = \int_1^2 (x+y)dy = (xy + \frac{y^2}{2}) \mid_{y=1}^2 = (2x+2) - (x + \frac{1}{2}) = x + \frac{3}{2}$$

and

$$I_2 = \int_0^1 G(x)dx = \int_0^1 (x + \frac{3}{2})dx = (\frac{x^2}{2} + \frac{3x}{2})\mid_0^1 = 2$$

In this example, $I_1 = I_2$

3.4.2 Corollary

If f is integrable on $[a,b] \times [c,d]$, then

$$\int_a^b dx \int_c^d f(x,y)dy = \int_c^d dy \int_a^b f(x,y)dx$$

provided that $\int_c^d f(x,y)dy$ exists for $a \le x \le b$ and $\int_a^b f(x,y)dx$ exists for $c \le y \le d$. In particular, these hypotheses hold if f is continuous on $[a,b] \times [c,d]$.

3.4.3 Theorem

Suppose that f is integrable on $R = [a,b] \times [c,d]$ and $F(y) = \int_a^b f(x,y)dx$ exists for each y in [c,d]. Then F is integrable on [c,d] and

$$\int_c^d F(y)dy = \int_R f(x,y)d(x,y).$$

That is,

$$\int_c^d dy \int_a^b f(x,y)dx = \int_R f(x,y)d(x,y)$$

Proof : Let

$$P_1 : a = x_0 < x_1 < \cdots < x_r = b \quad and \quad P_2 : c = y_0 < y_1 < \cdots < y_s = d$$

be partitions of [a,b] and [c,d], and $P = P_1 \times P_2$. Suppose that

$$y_{j-1} \le \eta_j \le y_j \quad , \quad 1 \le j \le s$$

so

$$\sigma = \sum_{j=1}^{r} f(\eta_j)(y_j - y_{j-1})$$

is a typical Riemann sum of f over P_2. Since

$$F(\eta_j) = \int_a^b f(x,\eta_j)dx = \sum_{i=1}^{r} \int_{x_{i-1}}^{x} f(x,\eta_j)dx$$

$$\Longrightarrow M_{ij} = \sup\{f(x,y)|x_{i-1} \le x \le x_i, \quad y_{j-1} \le y \le y_j\}$$

and

$$m_{ij} = \inf\{f(x,y)|x_{i-1} \le x \le x_i, \quad y_{j-1} \le y \le y_j\}$$

then

$$\sum_{i=1}^{r} m_{ij}(x_i - x_{i-1}) \le f(\eta_j) \le \sum_{i=1}^{r} M_{ij}(x_i - x_{i-1})$$

Multiplying this by $y_j - y_{j-1}$ and summing from $j = 1$ to $j = s$ yields

$$\sum_{j=1}^{s}\sum_{i=1}^{r} m_{ij}(x_i - x_{i-1})(y_s - y_{s-1}) \le \sum_{j=1}^{s} f(\eta_j)(y_j - y_{j-1}) \le \sum_{j=1}^{s}\sum_{i=1}^{r} M_{ij}(x_i - x_{i-1})(y_i - y_{i-1})$$

which can be written as

$$s_f(P) \le \sigma \le S_f(P)$$

Since f is integrable on R, there is for each $\epsilon > 0$ a partition P of R such that $S_f(P) - s_f(P) < \epsilon$. It remains to verify that there is for each $\epsilon > 0$, a $\delta > 0$ such that

$$|\int_c^d F(y)dy - \sigma| < \epsilon \quad if \quad ||P|| < \delta$$

In consequence,

$$s_f(P) - \epsilon < \int_c^d F(y)dy < S_f(P) + \epsilon \quad , \quad ||P|| < \delta$$

$$\Longrightarrow \underline{\int}_R f(x,y)d(x,y) - \epsilon \le \int_c^d F(y)dy \le \overline{\int}_R f(x,y)d(x,y) + \epsilon$$

Since

$$\underline{\int}_R f(x,y)d(x,y) = \overline{\int}_R f(x,y)d(x,y)$$

and ϵ can be made arbitrarily small.
Then, by interchanging x and y in the theorem, we see that

$$\int_a^b dx \int_c^d f(x,y)dy = \int_R f(x,y)d(x,y)$$

4 Note

A more detailed note was submitted with the lecturer in charge of the course which can be requested for at the Department of Mathematics, University of Ibadan, Ibadan, Oyo State, Nigeria.