The Seiberg-Witten equations for vector fields

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Abstract

By analogy with the Seiberg-Witten equations, we define equations for a spinor and a vector field.

1 Recalls of differential geometry

The $Spin-C$-structures are reductions of a $SO(n).S^1$- fiber bundle to the group $Spin^C(n) = Spin(n) \times_{\{1,-1\}} S^1$. For a four-manifold it exists always a $Spin-C$-structure for the tangent fiber bundle [F].

The Dirac operator is defined over the $Spin-C$-structure with help of a connection A for the associated line bundle.

$$\mathcal{D}_A = \sum_i e_i.\nabla^A_{e_i}$$

with ∇^A the connection defined by the Levi-Civita connection and the connection A of the determinant fiber bundle of the $Spin-C$-structure.

The self-dual part of the curvature (which is a 2-form) of the connection A is considered:

$$\Omega^+_A$$

A self-dual 2-form with imaginary values, bound to a spinor $\psi \in S^+$ is also defined by [F]:

$$\omega(\psi)(X,Y) =< X.Y.\psi, \psi > + < X,Y > |\psi|^2$$

2 Recalls of the Seiberg-Witten equations

The Seiberg-Witten equations are the following ones [F] [M]:

1)

$$\mathcal{D}_A(\psi) = 0$$

2)

$$\Omega^+_A = -(1/4)\omega(\psi)$$

3 The SWX equations

3.1 Definition

By analogy with the usual Seiberg-Witten equations, we are tempted to define equations for a spinor ψ and a vector field X:
1)

$$\mathcal{D}_X(\psi) = (\mathcal{D} + iX)(\psi) = 0$$

2)

$$id(X^*)^+ = -(1/4)\omega(\psi)$$

with X^* the dual form of X, d is the differential of the forms. The first equation makes use of the Clifford multiplication. We call these two equations, the SWX equations.

3.2 The gauge group

The gaug group acts:

$$f.(X, \psi) = (X + i(df)^*, f\psi)$$

3.3 The moduli spaces

We verify that we can define the quotient of the solutions of the SWX equations by the gauge group, it is the moduli space.

References

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