Existence of solutions for a class of nonlinear fractional Langevin equations with boundary conditions on the half-line

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Abstract

In this work, we use the fixed point theorems, we investigate the existence and uniqueness of solutions for a class of fractional Langevin equations with boundary value conditions on an infinite interval.

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1 Introduction

... To be completed.

In this paper, we investigate the existence and uniqueness of solutions for the following fractional Langevin equations with boundary conditions

$$\begin{array}{l} D^{\alpha}\left(D^{\beta}+\lambda\right)y\left(t\right)=f\left(t,y\left(t\right)\right), \qquad t\in(0,+\infty),\\ y(0)=D^{\beta}y\left(0\right)=0,\\ \lim\limits_{t\to+\infty}D^{\alpha-1}y\left(t\right)=ay(\xi_1), \quad \lim\limits_{t\to+\infty}D^{\alpha+\beta-1}y\left(t\right)=by(\xi_2), \end{array} \tag{1}$$

where $1<\alpha\leqslant 2$ and $0<\beta\leqslant 1$, such that $1<\alpha+\beta\leqslant 2$, with $a,b\in\mathbb{R}$, $\xi_1,\xi_2\in\mathbb{R}^{+}$ and D^{α}, D^{β} are the Riemman-Liouville fractional derivative. Some new results are obtained by applying standard fixed point theorems.

2 Preliminaries

Definition 1 *[2] The Riemann-Liouville fractional integral of ordre $\alpha \in \mathbb{R}^+$.for a function $f \in L^1[a,b]$ is defined as*

$$\left(I^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{2}$$

with Γ is Gamma Euler function.

Definition 2 *[2] Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}^*$ where $n-1 < \alpha < n$, The Riemann-Liouville dirivative integral of ordre α.for a function $f \in L^1[a,b]$ is defined as*

$$D_a^{\alpha} f(t) = D^n I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \tag{3}$$

with $D^n = \frac{d^n}{dt^n}$.

Properties

Let $\delta > 0$ and $\beta > 0$, for all $f \in L^1[a,b]$, we have

$$I^{\delta} I^{\beta} f(t) = I^{\beta} I^{\delta} f(t) = I^{\delta+\beta} f(t) \tag{4}$$

$$I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \quad \beta > -1. \tag{5}$$

If $\beta > \delta > 0$ we have

$$D^{\delta} I^{\beta} f(t) = I^{\beta-\delta} f(t). \tag{6}$$

Lemma 3 *[2]Let $\alpha \in \mathbb{R}^+$ where $n-1 < \alpha \leqslant n$, wiht $n \in \mathbb{N}^*$. Then the differential equation $D^{\alpha} y(t) = 0$, has this general solution*

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}, \tag{7}$$

where $c_i \in \mathbb{R}$, with $i = 0,1,2,...,n$.

Lemma 4 *[2]Let $\alpha > 0$. Then*

$$I^{\alpha} D^{\alpha} y(t) = y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}, \tag{8}$$

where $c_i \in \mathbb{R}$, with $i = 0,1,2,...,n$, and $n-1 < \alpha \leqslant n$.

3 Main results

Lemma 5 *Let $h(t) \in C(\mathbb{R}^+, \mathbb{R})$, $1 < \alpha \leqslant 2$ and $0 < \beta \leqslant 1$, with $1 < \alpha+\beta \leqslant 2$. The following problem*

$$\begin{array}{l} D^{\alpha}\left(D^{\beta} + \lambda\right) y(t) = h(t), \qquad t \in (0, +\infty), \\ y(0) = D^{\beta} y(0) = 0, \\ \lim\limits_{t \to +\infty} D^{\alpha-1} y(t) = a y(\xi_1), \quad \lim\limits_{t \to +\infty} D^{\alpha+\beta-1} y(t) = b y(\xi_2), \end{array} \tag{9}$$

has equivalent to the fractional integral equation

$$\begin{aligned} y(t) &= -\lambda\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}y(s)\,ds + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}h(s)ds \\ &+\mu t^{\beta+\alpha-1}\int_0^{+\infty}h(s)ds + \frac{a\mu\lambda^2 t^{\beta+\alpha-1}}{\Gamma(\beta)}\int_0^{\xi_1}(\xi_1-s)^{\beta-1}y(s)\,ds \\ &-\frac{a\mu\lambda t^{\beta+\alpha-1}}{\Gamma(\alpha+\beta)}\int_0^{\xi_1}(\xi_1-s)^{\alpha+\beta-1}h(s)ds + \frac{b\mu\lambda t^{\beta+\alpha-1}}{\Gamma(\beta)}\int_0^{\xi_2}(\xi_2-s)^{\beta-1}y(s)\,ds \\ &-\frac{b\mu t^{\beta+\alpha-1}}{\Gamma(\alpha+\beta)}\int_0^{\xi_2}(\xi_2-s)^{\alpha+\beta-1}h(s)ds. \end{aligned} \tag{10}$$

where

$$\mu = \left[b\xi_2^{\beta+\alpha-1} + \lambda a\xi_1^{\beta+\alpha-1} - \Gamma(\alpha+\beta)\right]^{-1}. \tag{11}$$

Proof. We applied the operartor I^α on $D^\alpha\left(D^\beta+\lambda\right)y(t) = h(t)$, we get

$$\left(D^\beta+\lambda\right)y(t) = I^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \tag{12}$$

where $c_1, c_2 \in \mathbb{R}$,
by the boundary condition $y(0) = 0$ and $D^\beta y(0) = 0$ we have $c_2 = 0$, thus

$$D^\beta y(t) = -\lambda y(t) + I^\alpha h(t) + c_1 t^{\alpha-1}, \tag{13}$$

applied the operartor I^β

$$y(t) = -\lambda I^\beta y(t) + I^{\alpha+\beta}h(t) + c_1 I^\beta t^{\alpha-1} + c_3 t^{\beta-1}, \tag{14}$$

where $c_3 \in \mathbb{R}$
by the boundary condition $y(0) = 0$ we have $c_3 = 0$, therefore

$$y(t) = -\lambda I^\beta y(t) + I^{\alpha+\beta}h(t) + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}t^{\alpha+\beta-1}. \tag{15}$$

Applied the operator $D^{\alpha+\beta-1}$

$$\begin{aligned} D^{\alpha+\beta-1}y(t) &= -\lambda D^{\alpha+\beta-1}I^\beta y(t) + D^{\alpha+\beta-1}I^{\alpha+\beta}h(t) \\ &+c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}D^{\alpha+\beta-1}t^{\alpha+\beta-1}, \end{aligned} \tag{16}$$

which yields

$$D^{\alpha+\beta-1}y(t) = -\lambda D^{\alpha+\beta-1}I^\beta y(t) + Ih(t) + c_1\Gamma(\alpha). \tag{17}$$

We have

$$\begin{aligned} D^{\alpha+\beta-1} I^{\beta} y(t) &= \frac{d}{dt} I^{1-(\alpha+\beta-1)} I^{\beta} y(t) \\ &= \frac{d}{dt} I^{2-\alpha} y(t) \\ &= \frac{d}{dt} I^{1-(\alpha-1)} y(t) \\ &= D^{\alpha-1} y(t). \end{aligned} \tag{18}$$

Substituting (18) into (17), we obtain

$$D^{\alpha+\beta-1} y(t) = -\lambda D^{\alpha-1} y(t) + Ih(t) + c_1 \Gamma(\alpha), \tag{19}$$

which yields

$$\lim_{t\to+\infty} D^{\alpha+\beta-1} y(t) = -\lambda \lim_{t\to+\infty} D^{\alpha-1} y(t) + \lim_{t\to+\infty} Ih(t) + c_1 \Gamma(\alpha). \tag{20}$$

By the eq (15) we have

$$y(\xi_1) = -\lambda I^{\beta} y(\xi_1) + I^{\alpha+\beta} h(\xi_1) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \xi_1^{\beta+\alpha-1}, \tag{21}$$

and

$$y(\xi_2) = -\lambda I^{\beta} y(\xi_2) + I^{\alpha+\beta} h(\xi_2) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \xi_2^{\beta+\alpha-1}. \tag{22}$$

Using the boundary conditions $\lim_{t\to+\infty} D^{\alpha-1} y(t) = ay(\xi_1)$, $\lim_{t\to+\infty} D^{\alpha+\beta-1} y(t) = by(\xi_2)$, and substituting (21) and (22) into (20)

$$\begin{aligned} & -\lambda b I^{\beta} y(\xi_2) + b I^{\alpha+\beta} h(\xi_2) + c_1 \frac{b\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \xi_2^{\beta+\alpha-1} \\ = \; & \lambda^2 I^{\beta} y(\xi_1) - \lambda I^{\alpha+\beta} h(\xi_1) - c_1 \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \xi_1^{\beta+\alpha-1} \\ & + \lim_{t\to+\infty} Ih(t) + c_1 \Gamma(\alpha), \end{aligned} \tag{23}$$

we obtain

$$\begin{aligned} c_1 = \; & \frac{\mu\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \left\{ a\lambda^2 I^{\beta} y(\xi_1) - a\lambda I^{\alpha+\beta} h(\xi_1) \right. \\ & \left. + \lim_{t\to+\infty} Ih(t) + \lambda b I^{\beta} y(\xi_2) - b I^{\alpha+\beta} h(\xi_2) \right\}, \end{aligned} \tag{24}$$

where μ defined as in (11).
Substituting (24) into (15), we obtain

$$\begin{aligned} y(t) = \; & -\lambda I^{\beta} y(t) + I^{\alpha+\beta} h(t) \\ & + \mu t^{\beta+\alpha-1} \left\{ a\lambda^2 I^{\beta} y(\xi_1) - a\lambda I^{\alpha+\beta} h(\xi_1) \right. \\ & \left. + \lim_{t\to+\infty} Ih(t) + \lambda b I^{\beta} y(\xi_2) - b I^{\alpha+\beta} h(\xi_2) \right\}. \end{aligned} \tag{25}$$

Therefore

$$
\begin{aligned}
y(t) = & -\frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} y(s)\, ds + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} h(s) ds \\
& +\mu t^{\beta+\alpha-1}\int_0^{+\infty} h(s) ds + \frac{a\mu\lambda^2 t^{\beta+\alpha-1}}{\Gamma(\beta)}\int_0^{\xi_1} (\xi_1 - s)^{\beta-1} y(s)\, ds \\
& -\frac{a\mu\lambda t^{\beta+\alpha-1}}{\Gamma(\alpha+\beta)}\int_0^{\xi_1} (\xi_1 - s)^{\alpha+\beta-1} h(s) ds + \frac{b\mu\lambda t^{\beta+\alpha-1}}{\Gamma(\beta)}\int_0^{\xi_2} (\xi_2 - s)^{\beta-1} y(s)\, ds \\
& -\frac{b\mu t^{\beta+\alpha-1}}{\Gamma(\alpha+\beta)}\int_0^{\xi_2} (\xi_2 - s)^{\alpha+\beta-1} h(s) ds. \qquad (26)
\end{aligned}
$$

The proof is complete ■

Consider the space defined by

$$
E = \left\{ y \in C(\mathbb{R}^+, \mathbb{R}),\ \sup_{t \geqslant 0} \frac{y(t)}{1 + t^{\beta+\alpha-1}} \text{ exist} \right\} \qquad (27)
$$

and with the norm

$$
\|y\|_E = \sup_{t \geqslant 0} \frac{|y(t)|}{1 + t^{\beta+\alpha-1}}. \qquad (28)
$$

Lemma 6 *[1] The space $(E, \|.\|_E)$ is Banach space*

We define the operator $T : E \to E$ by

$$
\begin{aligned}
Ty(t) = & -\frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, y(s))\, ds + I^{\alpha+\beta}\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} f(s, y(s))\, ds \\
& +\mu t^{\beta+\alpha-1}\int_0^{+\infty} f(s, y(s))\, ds + \frac{a\mu\lambda^2 t^{\beta+\alpha-1}}{\Gamma(\beta)}\int_0^{\xi_1} (\xi_1 - s)^{\beta-1} f(s, y(s))\, ds \\
& -\frac{a\mu\lambda t^{\beta+\alpha-1}}{\Gamma(\alpha+\beta)}\int_0^{\xi_1} (\xi_1 - s)^{\alpha+\beta-1} f(s, y(s))\, ds + \frac{b\mu\lambda t^{\beta+\alpha-1}}{\Gamma(\beta)}\int_0^{\xi_2} (\xi_2 - s)^{\beta-1} f(s, y(s))\, ds \\
& -\frac{b\mu t^{\beta+\alpha-1}}{\Gamma(\alpha+\beta)}\int_0^{\xi_2} (\xi_2 - s)^{\alpha+\beta-1} f(s, y(s))\, ds, \qquad (29)
\end{aligned}
$$

where μ defined as in (11).

To be completed.

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[3] To be completed.