Una serie trigonométrica y la constante de Catalan

Edgar Valdebenito

18-09-2018 16:09:04

Resumen

En esta nota recordamos una serie trigonométrica
que involucra a la constante de Catalan

1 Introducción

La constante de Catalan (Eugène Charles Catalan , 1814-1894) usualmente denotada por G , se define por la serie:

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965\ldots \tag{1}$$

Algunas representaciones alternativas son:

$$G = \int_0^1 \frac{\tan^{-1} x}{x} dx \tag{2}$$

$$G = \int_1^{\infty} \frac{\ln x}{1+x^2} dx \tag{3}$$

$$G = \frac{\pi}{8} \ln\left(2+\sqrt{3}\right) + \frac{3}{2} \int_0^{2-\sqrt{3}} \frac{\tan^{-1} x}{x} dx \tag{4}$$

$$G = \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} dx \tag{5}$$

$$G = \frac{1}{2} \int_0^{\infty} \frac{x}{\cosh x} dx \tag{6}$$

En esta nota recordamos una serie trigonométrica que involucra a la constante G .

2 Una fórmula previa

Para $|a|<1, |x|<\pi$, se tiene:

$$2\tan^{-1} a + a\ln\left(1+a^2\right) = = 2a + \sum_{n=1}^{\infty} \frac{\left(2\cos x\right)^n a^{n+1}}{n\left(n+1\right)} \, {}_2F_1\left(n, \frac{n+1}{2}; \frac{n+3}{2}; -a^2\right) - 2\sum_{n=1}^{\infty} \frac{a^{n+1}\cos\left(nx\right)}{n\left(n+1\right)} \quad (7)$$

La función ${}_2F_1\left(a,b;c;x\right)$ es la hipergeometrica usual. La fórmula (7) se puede obtener por distintas teorías alternativas (Ej. Series de Fourier).

Una forma alternativa de la fórmula (7) es:

$$2\tan^{-1} a + a\ln\left(1+a^2\right) = = 2a + \sum_{n=0}^{\infty} \frac{a^{n+2}}{n+2} \sum_{k=0}^{[n/2]} \binom{n-k}{k} \frac{(-1)^k \left(2\cos x\right)^{n-2k+1}}{n-2k+1} - 2\sum_{n=1}^{\infty} \frac{a^{n+1}\cos\left(nx\right)}{n\left(n+1\right)} \quad (8)$$

donde $\left[x\right]$ representa la parte entera de x .

3 Una serie trigonométrica y la constante de Catalan

De las fórmulas (4) y (8) se tiene:

$$\frac{4}{3}G + \frac{\pi}{6}\left(1-\ln\left(2+\sqrt{3}\right)\right) - \left(8-4\sqrt{3}\right) + \left(4-2\sqrt{3}\right)\ln 2 + \left(2-\sqrt{3}\right)\ln\left(2-\sqrt{3}\right) = = \sum_{n=0}^{\infty} \frac{\left(2-\sqrt{3}\right)^{n+2}}{\left(n+2\right)^2} \sum_{k=0}^{[n/2]} \binom{n-k}{k} \frac{(-1)^k \left(2\cos x\right)^{n-2k+1}}{n-2k+1} - 2\sum_{n=1}^{\infty} \frac{\left(2-\sqrt{3}\right)^{n+1}\cos\left(nx\right)}{n\left(n+1\right)^2} \quad (9)$$

donde $|x|<\pi$.

Ejemplos: $x=0, x=\pi/2, x=\pi/3$

$$\frac{4}{3}G + \frac{\pi}{6}\left(1-\ln\left(2+\sqrt{3}\right)\right) - \left(8-4\sqrt{3}\right) + \left(4-2\sqrt{3}\right)\ln 2 + \left(2-\sqrt{3}\right)\ln\left(2-\sqrt{3}\right) = = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\left(2\left(2-\sqrt{3}\right)\right)^{n+2}}{\left(n+2\right)^2} \sum_{k=0}^{[n/2]} \binom{n-k}{k} \frac{(-1)^k 2^{-2k}}{n-2k+1} - 2\sum_{n=1}^{\infty} \frac{\left(2-\sqrt{3}\right)^{n+1}}{n\left(n+1\right)^2} \quad (10)$$

$$\frac{4}{3}G + \frac{\pi}{6}\left(1-\ln\left(2+\sqrt{3}\right)\right) - \left(8-4\sqrt{3}\right) + \left(4-2\sqrt{3}\right)\ln 2 + \left(2-\sqrt{3}\right)\ln\left(2-\sqrt{3}\right) = = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(2-\sqrt{3}\right)^{2n+1}}{n\left(2n+1\right)^2} \quad (11)$$

$$\frac{4}{3}G+\frac{\pi}{6}\left(1-\ln\left(2+\sqrt{3}\right)\right)-\left(8-4\sqrt{3}\right)+\left(4-2\sqrt{3}\right)\ln 2+\left(2-\sqrt{3}\right)\ln\left(2-\sqrt{3}\right)=$$

$$=\sum_{n=0}^{\infty}\frac{\left(2-\sqrt{3}\right)^{n+2}}{\left(n+2\right)^{2}}\sum_{k=0}^{[n/2]}\binom{n-k}{k}\frac{\left(-1\right)^{k}}{n-2k+1}-\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}\left(2-\sqrt{3}\right)^{3n-1}}{\left(3n-2\right)\left(3n-1\right)^{2}} \tag{12}$$

$$-\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n}\left(2-\sqrt{3}\right)^{3n}}{\left(3n-1\right)\left(3n\right)^{2}}-2\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n}\left(2-\sqrt{3}\right)^{3n+1}}{\left(3n\right)\left(3n+1\right)^{2}}$$

Referencias

1. Adamchik, V.: 32 representations for Catalan's constant. Wolfram Research, Inc. (victor@wolfram.com) , available at website http://www.wolfram.com/~victor/articles/catalan/catalan.html.
2. Berndt, B.C.: Ramanujan's Notebooks: Part I, Springer-Verlag , 1985.
3. Berndt, B.C.: Ramanujan's Notebooks: Part II, Springer-Verlag , 1989.
4. Berndt, B.C.: Ramanujan's Notebooks: Part III, Springer-Verlag , 1991.
5. Boros, G. and Moll, V.H.: Irresistible Integrals, Cambridge Univ. Press, 2004.
6. Borwein, J.M. and Borwein, P.B.: Pi and the AGM , Wiley-Interscience, John Wiley & Sons. , Toronto , 1987 .
7. Bradley, D.M.: Representations of Catalan's constant, 2001 , at http://germain.umemat.maine.edu/faculty/bradley/papers/c1.ps .
8. Fee, G.J.: Computation of Catalan's constant using Ramanujan's formula, ISSAC ' 90 . (Proc. Internat. Symp. Symbolic and Algebraic Computation, Aug. 1990) ACM Press – Addison Wesley , 1990 .
9. Lord, N.: Intriguing integrals: an Euler- inspired odyssey, Math. Gazette 91, 2007, 415-427.
10. Ramanujan, S.: On the integral $\int_0^x \frac{\tan^{-1} t}{t}\,dt$, Journal of the Indian Mathematical Society , VII (1915) , pp. 93 – 96.
11. Valdebenito, E.: Question 445: Catalan's Constant and Some Integration Questions. http://vixra.org/pdf/1804.0016v1pdf .