Question 477: Some Mathematical Formulas

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Abstract. This note presents a collection of mathematical formulas

1. Introduction

Notation:

- The number pi:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.1415... \quad (1)$$

- The Catalan constant G :

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159... \quad (2)$$

- The Euler-Mascheroni constant γ :

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n\right) = 0.5772... \quad (3)$$

- The hypergeometric function:

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad ,|x|<1 \quad (4)$$

- The Fibonacci numbers:

$$F_{n+2} = F_{n+1} + F_n \quad ,F_1 = F_2 = 1 \quad (5)$$

- $z \in \mathbb{C}, z = x + i\,y; x, y \in \mathbb{R}, i = \sqrt{-1}; x = \mathrm{Re}(z), y = \mathrm{Im}(z)$.

2. Formulas

Entry 1.

$$\pi\sqrt{3} = 4\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-12)^{-n}}{4n+1} \, {}_2F_1\left(1, n+\frac{1}{2}; 2n+\frac{3}{2}; \frac{2}{3}\right) \quad (6)$$

$$\pi\sqrt{3}=4\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-36)^{-n}}{4n+1}\,{}_2F_1\left(n+1,2n+\frac{1}{2};2n+\frac{3}{2};\frac{2}{3}\right) \tag{7}$$

$$\frac{\pi}{\sqrt{3}}=2\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-36)^{-n}}{4n+1}\,{}_2F_1\left(n+\frac{1}{2},2n+\frac{1}{2};2n+\frac{3}{2};-\frac{2}{3}\right) \tag{8}$$

$$\pi\sqrt{5}=6\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-60)^{-n}}{4n+1}\,{}_2F_1\left(1,n+\frac{1}{2};2n+\frac{3}{2};\frac{2}{5}\right) \tag{9}$$

$$\pi\sqrt{5}=6\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-100)^{-n}}{4n+1}\,{}_2F_1\left(n+1,2n+\frac{1}{2};2n+\frac{3}{2};\frac{2}{5}\right) \tag{10}$$

$$\frac{\pi}{\sqrt{5}}=2\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-60)^{-n}}{4n+1}\,{}_2F_1\left(1,n+1;2n+\frac{3}{2};-\frac{2}{3}\right) \tag{11}$$

Entry 2.

$$2G-\ln 2+\frac{\pi}{2}+1=\int_0^{\infty}e^{-\sqrt{x}}\sqrt{4-e^{-2\sqrt{x}}}\,dx-\int_0^{\left(\ln\sqrt{2}\right)^2}e^{\sqrt{x}}\sqrt{4-e^{2\sqrt{x}}}\,dx \tag{12}$$

Entry 3.

$$\frac{\pi^2}{12}=\int_0^1\tanh\left(\frac{\pi}{2}\sqrt{\frac{1}{x}-1}\right)dx \tag{13}$$

$$\frac{\pi^2}{24}=\int_0^{\infty}\frac{x}{\left(1+x^2\right)^2}\tanh\left(\frac{\pi x}{2}\right)dx \tag{14}$$

Entry 4.

$$\ln 2=\int_0^u\left(\cosh^{-1}f(x)-\cosh^{-1}g(x)\right)dx \tag{15}$$

where

$$u=\frac{1}{\phi}\sqrt{\frac{\phi-1}{\phi+1}}\quad,\phi=\frac{1+\sqrt{5}}{2} \tag{16}$$

$$f(x)=-\frac{1}{3}-2\sqrt{\frac{1}{9}+\frac{1}{3x^2}}\,\cos\left(\frac{2\pi}{3}+\frac{1}{3}\cos^{-1}\frac{x\left(18+x^2\right)}{\left(3+x^2\right)^{3/2}}\right) \tag{17}$$

$$g(x)=-\frac{1}{3}-2\sqrt{\frac{1}{9}+\frac{1}{3x^2}}\,\cos\left(\frac{4\pi}{3}+\frac{1}{3}\cos^{-1}\frac{x\left(18+x^2\right)}{\left(3+x^2\right)^{3/2}}\right) \tag{18}$$

Entry 6.

$$\pi-2=\int_0^1\ln\left(2-x^2+2\sqrt{1-x^2}\right)dx=\int_0^{2\ln 2}\sqrt{2e^{x/2}-e^{2x}}\,dx \tag{19}$$

$$\frac{\pi-2}{2\ln 2}=\int_0^1 \sqrt{2^{x+1}-2^{2x}}\,dx \tag{20}$$

Entry 7. If $k\in\mathbb{N}$, then

$$\pi=2\sqrt{3}\int_0^1 \psi_k(x)e^{-\sum_{n=1}^{k}a_n x^{2n}}\,dx \tag{21}$$

where

$$a_n=\frac{(-1)^{n-1}3^{-n}}{n} \quad ,n\in\mathbb{N} \tag{22}$$

$$\psi_k(x)=\sum_{n=0}^{\infty}3^{-n}f_n\,x^{2n} \quad ,k\in\mathbb{N} \tag{23}$$

$$f_{n+k+1}=\frac{(-1)^{n+k+1}}{n+k+1}\sum_{m=0}^{n}(-1)^m f_m \quad ,f_0=1,\ f_1=f_2=\ldots=f_k=0,n\in\mathbb{N}\cup\{0\} \tag{24}$$

Example $k=1$:

$$\pi=2\sqrt{3}\int_0^1 \psi_1(x)e^{-x^2/3}\,dx \tag{25}$$

$$\psi_1(x)=1+\sum_{n=2}^{\infty}3^{-n}f_n\,x^{2n}=1+\frac{3^{-2}x^4}{2}-\frac{3^{-3}x^6}{3}+\ldots \tag{26}$$

$$f_{n+2}=\frac{(-1)^n}{n+2}\sum_{m=0}^{n}(-1)^m f_m \quad ,f_0=1,\ f_1=0,n\in\mathbb{N}\cup\{0\} \tag{27}$$

Example $k=2$:

$$\pi=2\sqrt{3}\int_0^1 \psi_2(x)e^{-\frac{x^2}{3}+\frac{x^4}{18}}\,dx \tag{28}$$

$$\psi_2(x)=1+\sum_{n=3}^{\infty}3^{-n}f_n\,x^{2n}=1-\frac{3^{-3}x^6}{3}+\ldots \tag{29}$$

$$f_{n+3}=\frac{(-1)^{n+1}}{n+3}\sum_{m=0}^{n}(-1)^m f_m \quad ,f_0=1,\ f_1=f_2=0,n\in\mathbb{N}\cup\{0\} \tag{30}$$

Entry 8.

$$\begin{aligned}\frac{\pi}{2\sqrt{3}}-\frac{8}{9}=\sum_{n=2}^{\infty}\frac{(-1)^n 3^{-n}}{2n+1}\,{}_2F_1\left(1,1+\frac{1}{2n};2+\frac{1}{2n};(-3)^{-n}\right)\\ -\sum_{n=2}^{\infty}\frac{(-1)^n 3^{-n^2}}{2n^2+1}\,{}_2F_1\left(1,n+\frac{1}{2n};n+1+\frac{1}{2n};(-3)^{-n}\right)\\ -\sum_{n=2}^{\infty}\frac{3^{-n^2-n}}{2n^2+2n+1}\,{}_2F_1\left(1,n+1+\frac{1}{2n};n+2+\frac{1}{2n};(-3)^{-n}\right)\end{aligned} \tag{31}$$

Entry 9. If $0<a<1$, then

$$\int_0^\infty\int_0^\infty \frac{e^{-x^2y}\sinh(ay)}{\cosh y}\,dxdy=\frac{\pi}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{\sqrt{2n+1-a}}-\frac{1}{\sqrt{2n+1+a}}\right) \tag{32}$$

Entry 10.

$$\int_0^\infty \frac{(1+x)e^{-x}}{2+2x+x^2}\,dx=-e\left(\left(\gamma+\frac{\ln 2}{2}\right)\cos 1-\frac{\pi}{4}\sin 1\right)+e\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,n!}\operatorname{Re}\left(e^i(1+i)^n\right) \tag{33}$$

$$\int_0^\infty \frac{e^{-x}}{2+2x+x^2}\,dx=e\left(\left(\gamma+\frac{\ln 2}{2}\right)\sin 1+\frac{\pi}{4}\cos 1\right)-e\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,n!}\operatorname{Im}\left(e^i(1+i)^n\right) \tag{34}$$

Entry 11.

$$\int_0^{\sqrt{e}} \sin^{-1}\left(2W(x/2)\right)dx=\frac{\pi\sqrt{e}}{2}-\sum_{n=0}^{\infty}\binom{2n}{n}^{-2}\frac{1}{(n!)^2(2n+1)}-\frac{\pi}{8}\sum_{n=0}^{\infty}\frac{2^{-4n}}{n!(n+1)!} \tag{35}$$

where $W(x)$ is the Lambert function.

Entry 12.

$$\frac{\pi}{3\sqrt{3}}=\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{1}{2^{k-n}(2n+1)\binom{2^{k-n+1}(2n+1)}{2^{k-n}(2n+1)}} \tag{36}$$

Entry 13.

$$\pi+4-5\sqrt{2}=\int_{1/4}^{(2+\sqrt{2})^{-1}}\frac{\sqrt{16x^3-20x^2+8x-1}}{x^2}\,dx \tag{37}$$

Entry 14. If $n\in\mathbb{N}$, then

$$\int_0^1\left(\sqrt{1+x}+\sqrt{1-x}\right)^{2n}dx=$$
$$=2^{n-1}\left\{\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k+2}{k+1}^{-1}\frac{2^{2k+2}}{k+1}+\pi\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}\binom{2k+2}{k+1}2^{-2k-2}\right\} \tag{38}$$

$$\int_0^1\left(\sqrt{1+x}-\sqrt{1-x}\right)^{2n}dx=$$
$$=2^{n-1}\left\{\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k+2}{k+1}^{-1}\frac{2^{2k+2}}{k+1}-\pi\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}\binom{2k+2}{k+1}2^{-2k-2}\right\} \tag{39}$$

Entry 15.

$$\int_0^1 \sin^{-1}\left(\sqrt{\frac{2+x^2-x\sqrt{8+x^2}}{2}}\right)dx=\frac{\sqrt{\pi}}{2}\left(\frac{\left(\Gamma(1/4)\right)^2}{\pi\sqrt{8}}-\frac{\pi\sqrt{8}}{\left(\Gamma(1/4)\right)^2}\right) \tag{40}$$

$$\int_0^1 \cos^{-1}\left(\sqrt{\frac{x\sqrt{8+x^2}-x^2}{2}}\right)dx=\frac{\sqrt{\pi}}{2}\left(\frac{\left(\Gamma(1/4)\right)^2}{\pi\sqrt{8}}-\frac{\pi\sqrt{8}}{\left(\Gamma(1/4)\right)^2}\right) \tag{41}$$

Entry 16. If F_n is the Fibonacci number, then

$$\pi=6\sum_{n=2}^{\infty}\tan^{-1}\left(\frac{\left(\sqrt{3}\right)^{F_{n+1}}-\left(\sqrt{3}\right)^{F_n}}{\left(\sqrt{3}\right)^{F_{n+2}}+1}\right) \tag{42}$$

$$\pi=8\sum_{n=2}^{\infty}\tan^{-1}\left(\frac{\left(\sqrt{2}-1\right)^{F_n}-\left(\sqrt{2}-1\right)^{F_{n+1}}}{1+\left(\sqrt{2}-1\right)^{F_{n+2}}}\right) \tag{43}$$

$$\pi=12\sum_{n=2}^{\infty}\tan^{-1}\left(\frac{\left(2-\sqrt{3}\right)^{F_n}-\left(2-\sqrt{3}\right)^{F_{n+1}}}{1+\left(2-\sqrt{3}\right)^{F_{n+2}}}\right) \tag{44}$$

Entry 17. If F_n is the Fibonacci number, then

$$\frac{\pi}{6}+\sum_{n=1}^{\infty}\tan^{-1}\left(\left(\frac{1}{\sqrt{3}}\right)^{F_{n+2}}\right)=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\left(\sqrt{3}\right)^{F_{2n}}+\left(\sqrt{3}\right)^{F_{2n+1}}}{\left(\sqrt{3}\right)^{F_{2n+2}}-1}\right) \tag{45}$$

Entry 18. If F_n is the Fibonacci number, then

$$\frac{\pi}{6}=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\left(\sqrt{3}\right)^{F_{2n+1}}-\left(\sqrt{3}\right)^{F_{2n}}}{\left(\sqrt{3}\right)^{F_{2n+2}}+1}\right)+\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\left(\sqrt{3}\right)^{F_{2n+2}}-\left(\sqrt{3}\right)^{F_{2n+1}}}{\left(\sqrt{3}\right)^{F_{2n+3}}+1}\right) \tag{46}$$

Entry 19. If F_n is the Fibonacci number, then

$$\frac{\pi}{6}+\sum_{n=1}^{\infty}(-1)^n\tan^{-1}\left(\left(\frac{1}{\sqrt{3}}\right)^{F_{n+2}}\right)=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\left(\sqrt{3}\right)^{F_{2n}+1}-\left(\sqrt{3}\right)^{F_{2n}}}{\left(\sqrt{3}\right)^{F_{2n+2}}+1}\right) \tag{47}$$

Entry 20. If $\sinh u=\frac{1}{6}\left(54+6\sqrt{129}\right)^{1/3}-2\left(54+6\sqrt{129}\right)^{-1/3}$, then

$$\pi=8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left(\tanh u\right)^{2n-1}\cosh\left((2n-1)u\right) \tag{48}$$

Entry 21. If $\sinh u=\frac{1}{3}\sqrt[3]{6}\,\sqrt[6]{3}-\frac{1}{18}\sqrt[3]{6^2}\,\sqrt[6]{3^5}$, then

$$\pi=12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left(\tanh u\right)^{2n-1}\cosh\left((2n-1)u\right) \tag{49}$$

Entry 22. If

$$\sinh u=\frac{1}{6}\left(-54+54\sqrt{2}+6\sqrt{291-162\sqrt{2}}\right)^{1/3}-2\left(-54+54\sqrt{2}+6\sqrt{291-162\sqrt{2}}\right)^{-1/3}$$

then

$$\pi = 16\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}(\tanh u)^{2n-1}\cosh((2n-1)u) \tag{50}$$

Entry 23. If

$$\sinh u = \frac{1}{6}\left(108-54\sqrt{3}+6\sqrt{615-324\sqrt{3}}\right)^{1/3} - 2\left(108-54\sqrt{3}+6\sqrt{615-324\sqrt{3}}\right)^{-1/3}$$

then

$$\pi = 24\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}(\tanh u)^{2n-1}\cosh((2n-1)u) \tag{51}$$

Entry 24. If $u^2(\cos(2u)-\sin(2u))+2u(\sin u+\cos u)\cosh u-1=0$, then

$$\pi = 8\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{u^{2n-1}\cosh((2n-1)u)\cos((2n-1)u)}{2n-1}+\frac{u^{2n}\cosh(2nu)\sin(2nu)}{2n}\right) \tag{52}$$

Remark: $u = 0.35993392348862...$.

Entry 25. If

$$a = \left(2-\sqrt{3}\right)\cos\left(\left(2-\sqrt{3}\right)\cos\left(\left(2-\sqrt{3}\right)\cos\left(\left(2-\sqrt{3}\right)...\right)\right)\right) \tag{53}$$

$$b = \left(\sqrt{2}-1\right)\cos\left(\left(\sqrt{2}-1\right)\cos\left(\left(\sqrt{2}-1\right)\cos\left(\left(\sqrt{2}-1\right)...\right)\right)\right) \tag{54}$$

$$c = \frac{1}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}\cos\left(\frac{1}{\sqrt{3}}...\right)\right)\right) \tag{55}$$

$$d = 1\cos(1\cos(1\cos(1...))) \tag{56}$$

then

$$\pi = 12\sum_{n=0}^{\infty}\frac{(-1)^n h_{2n}\, a^{2n+1}}{2n+1} = 8\sum_{n=0}^{\infty}\frac{(-1)^n h_{2n}\, b^{2n+1}}{2n+1} = \\ = 6\sum_{n=0}^{\infty}\frac{(-1)^n h_{2n}\, c^{2n+1}}{2n+1} = 4\sum_{n=0}^{\infty}\frac{(-1)^n h_{2n}\, d^{2n+1}}{2n+1} \tag{57}$$

where

$$h_0 = 1, h_1 = 0, h_2 = -\frac{1}{2}, h_3 = \frac{2}{3}, h_4 = -\frac{11}{24}, h_5 = \frac{2}{15}, ... \tag{58}$$

$$h_n = g_n - h_{n-1} - \sum_{k=2}^{n} h_{n-k} f_k \quad , n = 2,3,4,... \tag{59}$$

$$f_1 = 1, f_{2n+1} = 0, n\in\mathbb{N}, f_{2n} = \frac{1}{(2n)!}, n\in\mathbb{N} \tag{60}$$

$$g_{2n-1} = \frac{1}{(2n-1)!}, n\in\mathbb{N}, g_{2n} = 0, n\in\mathbb{N} \tag{61}$$

Entry 26.

$$\int_0^1 \frac{\tan^{-1}(3x)}{1+9x^2}\sin^{-1}\left(\frac{4x}{1+\sqrt{1+8x^2}}\right)dx =$$
$$= \frac{\pi}{6}\left(\frac{\left(\tan^{-1}3\right)^2}{2} - \sum_{n=1}^{\infty}\frac{2\left(\left(\sqrt{2}-1\right)\left(\sqrt{5}-2\right)\right)^{2n-1} + \left(3-\sqrt{8}\right)^{2n-1} + \left(9-\sqrt{80}\right)^{2n-1}}{(2n-1)^2}\right) \tag{62}$$

Entry 27.

$$\pi G = \int_0^{\pi} x\tan x\tan^{-1}(\cos x)\,dx \tag{63}$$

Entry 28.

$$\frac{\pi}{4} = \int_0^{\infty}\frac{\tanh x - x\,\mathrm{sech}^2 x}{x^2+\tanh^2 x}dx = \int_0^{\infty}\frac{\cosh x\sinh x - x}{x^2\cosh^2 x + \sinh^2 x}dx =$$
$$= \int_0^{\infty}\frac{\sinh(2x)-2x}{\left(1+x^2\right)\cosh(2x)+x^2-1}dx \tag{64}$$

Entry 29.

$$\int_0^{\pi}\frac{\left(1-\left(x^2-y^2\right)\cos^2 t\right)t\sin t}{1-2\left(x^2-y^2\right)\cos^2 t+\left(x^2+y^2\right)^2\cos^4 t}dt =$$
$$= \frac{\pi}{2\left(x^2+y^2\right)}\left\{\frac{x}{2}\ln\left(\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right) + y\left(\tan^{-1}\frac{y}{1+x}+\tan^{-1}\frac{y}{1-x}\right)\right\} \tag{65}$$

$$\int_0^{\pi}\frac{2xyt\sin t\cos^2 t}{1-2\left(x^2-y^2\right)\cos^2 t+\left(x^2+y^2\right)^2\cos^4 t}dt =$$
$$= \frac{\pi}{2\left(x^2+y^2\right)}\left\{x\left(\tan^{-1}\frac{y}{1+x}+\tan^{-1}\frac{y}{1-x}\right) - \frac{y}{2}\ln\left(\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right)\right\} \tag{66}$$

$$\int_0^{\pi}\frac{t\sin t}{1-2\left(x^2-y^2\right)\cos^2 t+\left(x^2+y^2\right)^2\cos^4 t}dt =$$
$$= \frac{\pi}{4}\left\{\frac{1}{2x}\ln\left(\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right) + \frac{1}{y}\left(\tan^{-1}\frac{y}{1+x}+\tan^{-1}\frac{y}{1-x}\right)\right\} \tag{67}$$

$$\int_0^{\pi} \frac{t\sin^3 t}{1-2\left(x^2-y^2\right)\cos^2 t+\left(x^2+y^2\right)^2\cos^4 t}\,dt =$$

$$= \frac{\pi}{4}\left\{\left(\frac{1}{2x}+\frac{1}{2x\left(x^2+y^2\right)}\right)\ln\left(\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right)+\left(\frac{1}{y}-\frac{1}{y\left(x^2+y^2\right)}\right)\left(\tan^{-1}\frac{y}{1+x}+\tan^{-1}\frac{y}{1-x}\right)\right\} \tag{68}$$

Entry 30.

$$\int_0^{\pi} \frac{\left(x-\left(x^3+xy^2\right)\cos^2 t\right)t\sin t}{1-2\left(x^2-y^2\right)\cos^2 t+\left(x^2+y^2\right)^2\cos^4 t}\,dt = \frac{\pi}{4}\ln\left(\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right) \tag{69}$$

$$\int_0^{\pi} \frac{\left(y+\left(x^2y+y^3\right)\cos^2 t\right)t\sin t}{1-2\left(x^2-y^2\right)\cos^2 t+\left(x^2+y^2\right)^2\cos^4 t}\,dt = \frac{\pi}{2}\left(\tan^{-1}\frac{y}{1+x}+\tan^{-1}\frac{y}{1-x}\right) \tag{70}$$

Entry 31.

$$\int_0^{\pi} \frac{\left(1+\left(2\sqrt{3}-3\right)\cos^2 t\right)t\sin t}{1+\left(4\sqrt{3}-6\right)\cos^2 t+\left(229-132\sqrt{3}\right)\cos^4 t}\,dt = \frac{\pi\left(5\left(7+5\sqrt{3}\right)\pi+6\left(4+\sqrt{3}\right)\ln 2\right)}{312} \tag{71}$$

$$\int_0^{\pi} \frac{t\sin t\cos^2 t}{1+\left(4\sqrt{3}-6\right)\cos^2 t+\left(229-132\sqrt{3}\right)\cos^4 t}\,dt = \frac{\pi\left(5\left(29+17\sqrt{3}\right)\pi-12\left(40+23\sqrt{3}\right)\ln 2\right)}{1248} \tag{72}$$

$$\int_0^{\pi} \frac{t\sin t}{1+\left(4\sqrt{3}-6\right)\cos^2 t+\left(229-132\sqrt{3}\right)\cos^4 t}\,dt = \frac{\pi\left(5\left(1+\sqrt{3}\right)\pi+12\left(2+\sqrt{3}\right)\ln 2\right)}{96} \tag{73}$$

$$\int_0^{\pi} \frac{\left(2+\cos^2 t\right)t\sin t}{1+\cos^2 t+\cos^4 t}\,dt = \frac{\pi\left(\pi\sqrt{3}+\ln 3\right)}{4} \tag{74}$$

$$\int_0^{\pi} \frac{t\sin t\cos^2 t}{1+\cos^2 t+\cos^4 t}\,dt = \frac{\pi\left(\pi\sqrt{3}-3\ln 3\right)}{12} \tag{75}$$

$$\int_0^{\pi} \frac{t\sin t}{1+\cos^2 t+\cos^4 t}\,dt = \frac{\pi}{4}\left(\frac{\pi}{\sqrt{3}}+\ln 3\right) \tag{76}$$

$$\int_0^{\pi} \frac{t\sin^3 t}{1+\cos^2 t+\cos^4 t}\,dt = \frac{\pi \ln 3}{2} \tag{77}$$

$$\int_0^{\pi} \frac{t\sin^3 t}{1+\cos^4 t}\,dt = \frac{\pi}{\sqrt{2}}\ln\left(\sqrt{2}+1\right) \tag{78}$$

$$\int_0^{\pi} \frac{t\sin t\left(2-\sin^2 t\right)}{1+\cos^2 t+\cos^4 t}\,dt = \int_0^{\pi} \frac{t\sin t\left(1+\cos^2 t\right)}{1+\cos^2 t+\cos^4 t}\,dt = \frac{\pi^2}{2\sqrt{3}} \tag{79}$$

$$\int_0^{\pi} \frac{t\sin^3 t}{9-2\cos^2 t+\cos^4 t}\,dt = \frac{\pi}{24}\left(4\sqrt{2}\ln\left(\sqrt{2}+1\right)-\pi\right) \tag{80}$$

$$\int_0^{\pi} \frac{t\sin^3 t}{7+\cos(4t)}\,dt = \frac{\pi}{8\sqrt{3}}\ln\left(2+\sqrt{3}\right) \tag{81}$$

$$\int_0^{\pi} \frac{t\sin t\left(8-\cos^2 t\right)}{81+4\cos^4 t}\,dt = \frac{\pi}{12}\ln\frac{17}{5} \tag{82}$$

$$\int_0^{\pi} \frac{t\sin t\left(2+\cos^2 t\right)}{4+\cos^4 t}\,dt = \frac{\pi}{2}\left(\frac{\pi}{4}+\tan^{-1}\frac{1}{3}\right) \tag{83}$$

Entry 32.

$$\begin{aligned}\int_1^{\infty} \frac{J_0^2(x)-e^{-x^2}\cos x}{x}\,dx &= \ln 2-\frac{\gamma}{2}+\frac{1}{4}\,{}_2F_2\left(1,1;\frac{3}{2},2;-\frac{1}{4}\right)\\ &\quad -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n}\left(\frac{1}{(2n-2k)!k!}-\frac{2^{-2n}}{\left((n-k)!k!\right)^2}\right)\end{aligned} \tag{84}$$

Remarks:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n}\frac{1}{k!(2n-2k)!} = -\sum_{n=1}^{\infty}\frac{2^{2n}}{n(2n)!}U\left(-n,\frac{1}{2},-\frac{1}{4}\right) \tag{85}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n}\frac{2^{-2n}}{\left(k!(n-k)!\right)^2} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{-2n}}{n(n!)^2}\binom{2n}{n} = \frac{1}{2}\,{}_3F_4\left(1,1,\frac{3}{2};2,2,2,2;-1\right) \tag{86}$$

- $U(a,b,z)$ is the confluent hypergeometric function:

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^{\infty} e^{-zt}t^{a-1}(1+t)^{b-a-1}\,dt \quad ,\mathrm{Re}(a)>0 \tag{87}$$

$$U(-n,b,z) = (-1)^n\sum_{k=0}^{n}\binom{n}{k}(b+k)_{n-k}(-z)^k \tag{88}$$

- $J_0(x)$ is the Bessel function.

Entry 33. If $|x|<\pi,|a|<1$, then

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{a^n\sin x}{1-a^n\cos x}\right)=\sum_{n=1}^{\infty}\frac{a^n}{1-a^n}\frac{\sin(nx)}{n} \tag{89}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\tan^{-1}\left(\frac{a^n\sin x}{1-a^n\cos x}\right)=\sum_{n=1}^{\infty}\frac{a^n}{1+a^n}\frac{\sin(nx)}{n} \tag{90}$$

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{2a^n\sin x}{1-a^{2n}}\right)=2\sum_{n=0}^{\infty}\frac{a^{2n+1}}{1-a^{2n+1}}\frac{\sin((2n+1)x)}{2n+1} \tag{91}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\tan^{-1}\left(\frac{2a^n\sin x}{1-a^{2n}}\right)=2\sum_{n=0}^{\infty}\frac{a^{2n+1}}{1+a^{2n+1}}\frac{\sin((2n+1)x)}{2n+1} \tag{92}$$

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{2a^n\cos x}{1-a^{2n}}\right)=2\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{1-a^{2n+1}}\frac{\cos((2n+1)x)}{2n+1} \tag{93}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\tan^{-1}\left(\frac{2a^n\cos x}{1-a^{2n}}\right)=2\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{1+a^{2n+1}}\frac{\cos((2n+1)x)}{2n+1} \tag{94}$$

Entry 34. If $f(k)=\int_0^{\pi}e^{-x}\sin x\ln(x+k\pi)\,dx \quad ,k=0,1,2,3,...$, then

$$\frac{\pi}{4}-\gamma-\frac{\ln 2}{2}=\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{-k\pi}f(k) \tag{95}$$

Entry 35. If $f(k)=\int_0^{\pi}\frac{\ln(x+k\pi)}{x+k\pi}\sin x\,dx \quad ,k=0,1,2,3,...$, then

$$-\pi\gamma=\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k) \tag{96}$$

Entry 36. If $f(k)=\int_0^{\pi}\frac{\ln(x+(k+1)\pi)}{x+(k+1)\pi}\sin x\,dx \quad ,k=0,1,2,3,...$, then

$$\pi\gamma=-2\int_0^{\pi}\frac{\ln x\sin x}{x}dx+\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k) \tag{97}$$

Entry 37. If $f(k)=\int_0^{\pi}\frac{\ln(x+k\pi)}{x+k\pi}\sin 2x\sin x\,dx \quad ,k=0,1,2,3,...$, then

$$-\ln 3\left(\gamma+\frac{\ln 3}{2}\right)=\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k) \tag{98}$$

Entry 38.

$$\frac{\pi}{3}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n 2^{-2n}}{2n+1}\operatorname{Im}\left(\left(\frac{1+i\sqrt{15}}{4}\right)^{2n+1}\right) \tag{99}$$

$$\frac{\pi}{6}=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n 2^{-2n}}{2n+1}\operatorname{Im}\left(\left(\frac{3+i\sqrt{7}}{4}\right)^{2n+1}\right) \tag{100}$$

Entry 39. If $0<a<b<1$, then

$$\frac{\pi}{4}\ln\frac{b}{a}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(b^{2n+1}-a^{2n+1}\right)}{(2n+1)^2}+$$
$$+\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left\{\ln\left(\left(\frac{1+a}{1+b}\right)^2\frac{b}{a}\right)+\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k}\left(\left(\frac{4b}{(1+b)^2}\right)^k-\left(\frac{4a}{(1+a)^2}\right)^k\right)\right\} \tag{101}$$

$$\frac{\pi}{4}\ln\frac{b}{a}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(b^{2n+1}-a^{2n+1}\right)}{(2n+1)^2}+$$
$$+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(n+1)}\left\{\left(\frac{1-a}{1+a}\right)^{2n+2}{}_2F_1\left(1,n+1;n+2;\left(\frac{1-a}{1+a}\right)^2\right)\right.$$
$$\left.-\left(\frac{1-b}{1+b}\right)^{2n+2}{}_2F_1\left(1,n+1;n+2;\left(\frac{1-b}{1+b}\right)^2\right)\right\} \tag{102}$$

$$\frac{\pi}{4}\ln\frac{b}{a}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(b^{2n+1}-a^{2n+1}\right)}{(2n+1)^2}+\sum_{n=0}^{\infty}\frac{1}{n+1}\left(\left(\frac{1-a}{1+a}\right)^{2n+2}-\left(\frac{1-b}{1+b}\right)^{2n+2}\right)\sum_{k=0}^{n}\frac{(-1)^k}{2k+1} \tag{103}$$

Entry 40.

$$\pi-2\tan^{-1}u=4\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_u^{\infty}\left(\tanh\frac{1}{2+2x^2}\right)^{2n+1}dx \tag{104}$$

$$\pi=24\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_{2+\sqrt{3}}^{\infty}\left(\tanh\frac{1}{2+2x^2}\right)^{2n+1}dx \tag{105}$$

Entry 41. If $0<z<1$, then

$$\frac{1}{\Gamma(z)}=\frac{1}{\pi}\sin\frac{z\pi}{2}\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^{\pi}\frac{\sin x}{(x+k\pi)^z}dx \tag{106}$$

Entry 42.

$$\frac{\pi}{4}=\int_0^1\frac{\sqrt{1-x^4}}{1+x^4}dx \tag{107}$$

$$\frac{\pi}{4\sqrt[4]{2}}=\int_0^1\sqrt[4]{\frac{1-x^2}{1+2x^2+\sqrt{1+8x^2}}}\,dx \tag{108}$$

Entry 43.

$$\frac{\pi}{4}+\frac{1}{2}=\int_0^1\cosh^{-1}\left(\frac{2}{\sqrt{1+8x}-1}\right)dx \tag{109}$$

$$\frac{\pi}{4}+\ln 2-\frac{1}{4}=\int_0^1 \ln\left(2+\sqrt{2}\sqrt{1-4x+\sqrt{1+8x}}\right)dx \tag{110}$$

Entry 44.

$$\frac{\pi}{4}-\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{2n(2n+1)+1}\right)=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^n\left(1-2^{-2n}\right)\zeta(2n+1)}{2n+1} \tag{111}$$

Entry 45.

$$\pi=8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\operatorname{Im}\left(\left(\sqrt{\frac{\sqrt{2}+1}{2}}-1+i\sqrt{\frac{\sqrt{2}-1}{2}}\right)^n\right) \tag{112}$$

$$\pi=8\sum_{n=1}^{\infty}\frac{2^{-n}}{n}\operatorname{Im}\left(\left(2-\sqrt{\sqrt{2}+1}+i\sqrt{\sqrt{2}-1}\right)^n\right) \tag{113}$$

Entry 46. If $k\in\mathbb{N}$, then

$$\pi=2^k\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{(k-1)-radicals}}\left(1+2\int_0^{\infty}\frac{\sinh x}{1+e^{2^{k+1}x}}dx\right) \tag{114}$$

$$\pi=2^k\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{(k-1)-radicals}}\left(1+2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^{2k+2}n^2-1}\right) \tag{115}$$

$$\pi=2^k\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{(k-1)-radicals}}\prod_{n=0}^{\infty}\prod_{m=0}^{n}\left(1-\frac{2^{-2(n-m+k)}}{(4m+2)^2}\right) \tag{116}$$

Entry 47. If $0<x<\tan^{-1}\left(1/\sqrt{2}\right)$, then

$$\pi=4\tan^{-1}\left(\frac{1-2\tan^2 x}{1+2\tan^2 x}\right)+8\sum_{n=0}^{\infty}\frac{(\sin x)^{2n+1}}{2n+1}\sin(2n+1)x \tag{117}$$

Entry 48. If $\tan^{-1}\left(1/\sqrt{2}\right)<x<\pi/2$, then

$$\pi=2\tan^{-1}\left(\frac{1}{2\tan^2 x}\right)+4\sum_{n=0}^{\infty}\frac{(\sin x)^{2n+1}}{2n+1}\sin(2n+1)x \tag{118}$$

Entry 49. If $0<x<\tan^{-1}\sqrt{2}$, then

$$\pi=2\tan^{-1}\left(\frac{\tan^2 x}{2}\right)+4\sum_{n=0}^{\infty}\frac{(-1)^n(\cos x)^{2n+1}}{2n+1}\cos(2n+1)x \tag{119}$$

Entry 50. If $\tan^{-1}\sqrt{2}<x<\pi/2$, then

$$\pi=4\tan^{-1}\left(\frac{1-2\cot^2 x}{1+2\cot^2 x}\right)+8\sum_{n=0}^{\infty}\frac{(-1)^n(\cos x)^{2n+1}}{2n+1}\cos(2n+1)x \tag{120}$$

Entry 51. If $0<x<\pi/2, 0<a<1, 0<b<1$, $b=\sqrt{1+\left(\frac{2a\sin^2 x}{1-a^2}\right)^2}-\frac{2a\sin^2 x}{1-a^2}$, then

$$\pi=4\sum_{n=0}^{\infty}\frac{a^{2n+1}+b^{2n+1}}{2n+1}\sin(2n+1)x \tag{121}$$

Entry 52. If $0<x<\pi/2, 0<a<1, 0<b<1$, $b=\sqrt{1+\left(\frac{2a\cos^2 x}{1-a^2}\right)}-\frac{2a\cos^2 x}{1-a^2}$, then

$$\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n\left(a^{2n+1}+b^{2n+1}\right)}{2n+1}\cos(2n+1)x \tag{122}$$

Entry 53.

$$G=2\left(\sqrt{2}-1\right)+2\sum_{n=1}^{\infty}\frac{\left(\sqrt{2}-1\right)^{2n+1}}{2n+1}\left(2\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}+\frac{(-1)^n}{2n+1}\right) \tag{123}$$

Entry 54.

$$G=2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1\frac{x^{2n}}{\left(1+\sqrt{1+x^2}\right)^{2n+1}}dx \tag{124}$$

Entry 55. If $0<x<\pi/2$, then

$$G=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^2(\sin x)^n}\,{}_2F_1(n,n;n+1;-\cot x) \tag{125}$$

Entry 56. If $0<x<\pi/2$, then

$$G=\tan x\ln(1+\cot x)+ +\sum_{n=2}^{\infty}\frac{\sin(nx)}{n(\cos x)^n}\left(\ln(1+\cot x)+\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{(-1)^k}{k}\left(1-(1+\cot x)^{-k}\right)\right) \tag{126}$$

Entry 57.

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}a_n\left(\frac{1}{\sqrt{5}-1}\right)^{2n+1}+\sum_{n=0}^{\infty}b_n\left(\sqrt{5}-1\right)^{3n+3} \tag{127}$$

$$a_n=\frac{(-1)^n}{2n+1}\quad,\quad b_n=\frac{(-1)^n}{2^{4n+4}}\sum_{k=0}^{[n/2]}\binom{n}{n-2k}\frac{(-1)^k}{2k+1}\quad,n=0,1,2,3,\ldots \tag{128}$$

$$\frac{\pi}{2}=\sum_{n=0}^{\infty}A_n\,\phi^{2n+1}+\sum_{n=0}^{\infty}B_n\,\phi^{-3n-3} \tag{129}$$

$$A_n=\frac{(-1)^n}{(2n+1)2^{2n}}\quad,\quad B_n=\frac{(-1)^n}{2^n}\sum_{k=0}^{[n/2]}\binom{n}{n-2k}\frac{(-1)^k}{2k+1}\quad,n=0,1,2,3,\ldots \tag{130}$$

Remark: $\phi=\dfrac{1+\sqrt{5}}{2}$.

Entry 58. If $N\in\mathbb{N}=\{1,2,3,...\}$, then

$$\pi+2\ln 2=\frac{2}{\sqrt{N+1}}\left(2+\ln\left(N+1\right)\right)+2\sum_{n=1}^{N}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)\ln\left(n+1\right)+ \\ +\sum_{n=1}^{N}\sum_{k=1}^{\infty}\frac{\left(-1\right)^{k-1}{}_2F_1\left(3/2,k+1;k+2;-1/n\right)}{k\left(k+1\right)n\sqrt{n}\left(n+1\right)^k}+\frac{2}{\sqrt{N+1}}\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}\left(N+1\right)^{-n}}{n\left(2n+1\right)} \tag{131}$$

Entry 59.

$$G=\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{\infty}\left(\tanh\left(\frac{x}{2\cosh x}\right)\right)^{2n+1}dx \tag{132}$$

Entry 60. If $N>0$, then

$$G=N^2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{1}\left(\tanh\left(\frac{x}{2\cosh\left(Nx\right)}\right)\right)^{2n+1}dx+ \\ +\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_N^{\infty}\left(\tanh\left(\frac{x}{2\cosh x}\right)\right)^{2n+1}dx \tag{133}$$

Entry 61.

$$\gamma=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n\,n!}-\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n}\int_1^{\infty}\left(x^{e^{-x}}-1\right)^n dx \tag{134}$$

$$\gamma=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n\,n!}-\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\left(1-x^{-e^{-x}}\right)^n dx \tag{135}$$

$$\gamma=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n\,n!}-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_1^{\infty}\left(\frac{x^{e^{-x}}-1}{x^{e^{-x}}+1}\right)^{2n+1}dx \tag{136}$$

Entry 62.

$$3\pi-1=\int_0^1\frac{16\sqrt{1+8x}+16-65x}{x+4\sqrt{x\sqrt{1+8x}+x-4x^2}}dx \tag{137}$$

Entry 63. If $u=\dfrac{1}{6}\left(108+12\sqrt{93}\right)^{1/3}-2\left(108+12\sqrt{93}\right)^{-1/3}$, then

$$\frac{\pi}{2}+u^2=\tan^{-1}u+\sqrt{u}\left(2-\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}u^{n+1}}{\left(n+1\right)\left(2n+3\right)}\right) \tag{138}$$

Entry 64.

$$\frac{\pi^2}{4}=\left(\ln\left(\sqrt{2}+1\right)\right)^2+2\int_{\ln\left(\sqrt{2}+1\right)}^{\infty}\ln\left(\frac{e^x+1}{e^x-1}\right)dx \tag{139}$$

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