Differentiation under the loop integral: A new method of renormalization in quantum field theory

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In the conventional approach of renormalization, divergent loop integrals are regulated and combined with counterterms to satisfy a set of renormalization conditions. While successful, the process of regularization is tedious and must be applied judiciously to obtain gauge-invariant results. In this Letter, I show that by recasting the renormalization conditions as the initial conditions of momentum-space differential equations for the loop amplitudes, the need for regularization disappears because the process of differentiating under the loop integrals renders them finite. I apply this approach to successfully renormalize scalar ϕ^4 theory and QED to one-loop order without requiring regularization or counterterms. Beyond considerable technical simplifications, the ability to perform renormalization without introducing a regulator or counterterms can lead to a more fundamental description of quantum field theory free of ultraviolet divergences.

INTRODUCTION

Divergences are considered an unavoidable part of quantum field theory [1–3]. Scattering amplitudes for processes involving one or more loops are evaluated by integrating over internal momenta, resulting in formally divergent quantities. To obtain finite physical results, divergent integrals are regulated by introducing an arbitrary regulator parameter, and then combined with counterterms in the process of renormalization to obtain finite quantities independent of the regulator. This process of regularization and renormalization has proved very successful, and forms the basis of the standard model [4–8].

Several regularization methods have been developed and applied [9–18]. They can be evaluated based on their convenience and preservation of symmetries. The simplest is cutoff regularization, which imposes an upper limit on the loop momentum [14]. While simple, cutoff regularization breaks translation invariance, making it difficult to apply Feynman parametrization. It is also difficult to maintain gauge invariance when imposing a cutoff on the gauge covariant derivative. Related methods such as Gaussian or higher-derivative cutoff can be gauge invariant [15, 18], but suffer from lack of translation invariance. Pauli-Villars regularization [9], which introduces a divergent integral with much larger mass, maintains translation and gauge invariance, but is not gauge covariant, so it cannot be applied to QCD [19]. The most common approach is dimensional regularization, in which the spacetime dimension is treated as a continuous parameter [10, 11]. Dimensional regularization maintains translation and gauge invariance, but is difficult to apply to dimension-specific quantities, such as the Dirac gamma matrices [12], and is insensitive to quadratic divergences, which are important for understanding scaling behavior [16, 20].

Beyond superficial technical distinctions, the need for regularization in any form raises doubt about the logical foundations of quantum field theory [21–23]. In this article, I address this issue head on by developing a method of renormalization that does not require regularization or counterterms, and preserves all properties of the original theory. The approach is based on differentiation under the integral and the fact that any loop integral can be rendered finite by taking a sufficient number of derivatives with respect to external momenta. This process leads to momentum-space ordinary differential equations for the amplitudes which can be readily integrated to obtain the original amplitude up to integration constants. By imposing initial conditions, traditionally referred to as renormalization conditions, the renormalized amplitude is obtained without introducing counterterms. I apply this method to reproduce the results of dimensional regularization for several common loop integrals, and then apply it to renormalize scalar ϕ^4 theory and QED without ever involving regularization or counterterms. The ability to evaluate loop effects without introducing counterterms can greatly simply the calculation of higher-order processes, and more importantly, provide a more fundamental description of quantum field theory free of ultraviolet divergences.

APPROACH

The approach derives from the observation that the degree of divergence of a loop integral can be reduced by differentiating with respect to the external momenta. For example, consider the following integral frequently encountered in one-loop calculations

$$I_1(\Delta)=\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2}, \tag{1}$$

where Wick rotation is assumed. This integral is formally divergent, with a degree of divergence of zero. However, differentiating with respect to Δ gives

$$\begin{aligned}\frac{dI_1(\Delta)}{d\Delta} &= \frac{d}{d\Delta}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2} \\ &= \int \frac{d^4\ell}{(2\pi)^4}\frac{\partial}{\partial\Delta}\frac{1}{(\ell^2-\Delta)^2} \\ &= \int \frac{d^4\ell}{(2\pi)^4}\frac{2}{(\ell^2-\Delta)^3} \\ &= -\frac{1}{16\pi^2}\frac{1}{\Delta}. \end{aligned} \tag{2}$$

This is a consequence of the Leibniz rule of differentiation under the integral [24], which is perfectly valid because the integration limits are independent of Δ. The original divergent loop integral has been transformed into a finite differential equation for $I_1(\Delta)$, which can be readily integrated to obtain

$$I_1(\Delta) = -\frac{1}{16\pi^2}\log\Delta + c_1, \tag{3}$$

where c_1 is an integration constant. Equation (3) is equivalent to the result obtained by dimensional regularization with the replacement

$$c_1 \to \frac{1}{16\pi^2}\left[\frac{2}{\epsilon} - \gamma + \log(4\pi)\right], \tag{4}$$

where $\epsilon = 4-d$ and γ is the Euler-Mascheroni constant. Thus, what are normally understood as counterterms become initial conditions in the new approach. Consider another divergent integral often encountered in one-loop mass renormalization

$$I_2(\Delta) = \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)}. \tag{5}$$

In this case, the degree of divergence is two, so to obtain a finite result we must differentiate twice to obtain

$$\begin{aligned}\frac{d^2 I_2(\Delta)}{d\Delta^2} &= \frac{d^2}{d\Delta^2}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)} \\ &= \int \frac{d^4\ell}{(2\pi)^4}\frac{\partial^2}{\partial\Delta^2}\frac{1}{(\ell^2-\Delta)} \\ &= \int \frac{d^4\ell}{(2\pi)^4}\frac{2}{(\ell^2-\Delta)^3} \\ &= -\frac{1}{16\pi^2}\frac{1}{\Delta}. \end{aligned} \tag{6}$$

Thus, $I_2(\Delta)$ satisfies a second-order differential equation with the solution

$$I_2(\Delta) = -\frac{1}{16\pi^2}\Delta\log\Delta + c_2\Delta + c_1, \tag{7}$$

which, up to integration constants, also agrees with the result obtained by dimensional regularization [3]. Lastly, consider the common divergent integral

$$I_3(\Delta) = \int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^2}, \tag{8}$$

which is also quadratically divergent. To render it finite, we must again differentiate twice to obtain

$$\frac{d^2 I_3(\Delta)}{d\Delta^2} = -\frac{1}{8\pi^2}\frac{1}{\Delta}, \tag{9}$$

which can be integrated to obtain

$$I_3(\Delta) = -\frac{1}{8\pi^2}\Delta\log\Delta + c_2\Delta + c_1, \tag{10}$$

again in agreement with the result obtained by dimensional regularization [3]. Having successfully reproduced several results from dimensional regularization, I now proceed to apply the method to renormalize scalar ϕ^4 theory and QED.

APPLICATION TO ϕ^4 THEORY

Consider scalar ϕ^4 theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{11}$$

In conventional quantum field theory, the coupling constant, mass, and propagator residue are fixed at a particular momentum scale by a set of renormalization conditions. In the new formalism, renormalization conditions become the initial conditions of the theory.

Wavefunction and mass renormalization are derived from the one-loop amplitude

$$-iM^2 = -i\frac{\lambda}{2}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2+m^2}. \tag{12}$$

Coupling constant or vertex renormalization is calculated from the two particle scattering amplitude

$$i\mathcal{M} = -i\lambda + (-i\lambda)^2[iV(s) + iV(t) + iV(u)], \tag{13}$$

where s, t, and u, are the Mandelstam variables, and

$$V(p^2) = -\frac{1}{2}\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2}, \tag{14}$$

where $\Delta = x(1-x)p^2 - m^2$. Requiring the propagator pole to be at m with unit residue, and setting $i\mathcal{M} = -i\lambda$ at zero momentum leads to the initial conditions

$$\begin{aligned} M^2(p^2)\big|_{p^2=m^2} = 0, \quad \frac{d}{dp^2}M^2(p^2)\big|_{p^2=m^2} &= 0, \\ i\mathcal{M}\big|_{s=4m^2,t=u=0} &= -i\lambda. \end{aligned} \tag{15}$$

Since $M^2(p^2)$ has two initial conditions, it obeys a second-order differential equation. From Eq. (12),

$$\begin{aligned} -i\frac{d^2}{dp^4}M^2 = -i\frac{\lambda}{2}\int \frac{d^4\ell}{(2\pi)^4}\frac{\partial^2}{\partial p^4}\frac{1}{\ell^2+m^2} \\ = 0, \end{aligned} \tag{16}$$

with the solution

$$M(p^2)=c_1+c_2p^2. \tag{17}$$

The first two conditions in Eq. (15) imply $c_1=c_2=0$ and $M(p^2)=0$. Thus, in agreement with the standard result, there is no wavefunction or mass renormalization at one-loop order in ϕ^4 theory.

From Eq. (2), the three initial conditions on $\mathcal{M}$ lead to three first-order differential equations of the form

$$\begin{aligned}\frac{dV(p^2)}{d\Delta}&=-\frac{1}{2}\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{\partial}{\partial\Delta}\frac{1}{(\ell^2-\Delta)^2}\\&=\frac{1}{32\pi^2}\int_0^1 dx\frac{1}{\Delta},\end{aligned} \tag{18}$$

with the solution

$$V(p^2)=\frac{1}{32\pi^2}\int_0^1 dx\log[m^2-x(1-x)p^2]+c_1. \tag{19}$$

Imposing the initial condition in Eq. (15) gives,

$$c_1=-(-i\lambda)^2[iV(4m^2)+i2V(0)]. \tag{20}$$

Thus, the renormalized amplitude is

$$\begin{aligned}i\mathcal{M}=-i\lambda-\frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\log\left(\frac{m^2-x(1-x)s}{m^2-x(1-x)4m^2}\right)\right.\\\left.+\log\left(\frac{m^2-x(1-x)t}{m^2}\right)+\log\left(\frac{m^2-x(1-x)u}{m^2}\right)\right],\end{aligned} \tag{21}$$

which agrees with the standard result obtained by dimensional regularization [2, 3]. This completes the one-loop renormalization of scalar ϕ^4 theory.

APPLICATION TO QED

I now apply the method to QED with the Lagrangian

$$\mathcal{L}=-\frac{1}{4}(F_{\mu\nu})^2+\bar{\psi}(i\partial\!\!\!/-m)\psi-e\bar{\psi}\gamma^\mu\psi A_\mu. \tag{22}$$

The renormalization conditions are

$$\begin{aligned}&\Sigma(p\!\!\!/)\big|_{p\!\!\!/=m}=0,\quad \frac{d}{dp\!\!\!/}\Sigma(p\!\!\!/)\Big|_{p\!\!\!/=m}=0,\\&\Pi(q^2)\big|_{q^2=0}=0,\quad \frac{d\Pi(q^2)}{dq^2}\Big|_{q^2=0}=0,\\&-ie\Gamma^\mu(q^2)\big|_{q^2=0}=-ie\gamma^\mu,\end{aligned} \tag{23}$$

where the first and second lines fix the mass and propagator residue of the electron and photon, respectively, and the last condition fixes the vertex coupling. The first two conditions involve the electron self energy, which after applying Feynman parametrization and dropping the term linear in ℓ takes the form [3]

$$-i\Sigma_2(p)=-e^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^2}\frac{-2xp\!\!\!/+4m}{(\ell^2-\Delta)^2}, \tag{24}$$

where $\Delta=-x(1-x)p^2+(1-x)m^2$. Since $\Sigma_2(p)$ has two initial conditions, it obeys a second order differential equation. Taking the second derivative,

$$\begin{aligned}-i\frac{d^2\Sigma_2(p)}{dp\!\!\!/^2}=-e^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\left\{\frac{16p\!\!\!/(1-x)x^2}{(\ell^2-\Delta)^3}\right.\\\left.+(4m-2xp\!\!\!/)\left[\frac{24p^2(1-x)^2x^2}{(\ell^2-\Delta)^4}-\frac{4(1-x)x}{(\ell^2-\Delta)^3}\right]\right\}.\end{aligned} \tag{25}$$

All integrals are rendered finite by differentiation. Integrating over ℓ,

$$\begin{aligned}-i\frac{d^2\Sigma_2(p)}{dp\!\!\!/^2}=-\frac{e^2}{8\pi^2}\int_0^1 dx\left\{-\frac{4p\!\!\!/(1-x)x^2}{\Delta}\right.\\\left.+(4m-2xp\!\!\!/)\left[\frac{2p^2(1-x)^2x^2}{\Delta^2}+\frac{(1-x)x}{\Delta}\right]\right\},\end{aligned} \tag{26}$$

and then solving for $\Sigma_2(p)$,

$$\begin{aligned}-i\Sigma_2(p)=-\frac{e^2}{8\pi^2}\int_0^1 dx\,\big[(xp\!\!\!/-2m)\log(m^2-xp^2)\\+c_1+c_2p\!\!\!/\big].\end{aligned} \tag{27}$$

Applying the initial conditions in Eq. (23),

$$\begin{aligned}c_1&=m(2-x)\log[m^2(1-x)]+m\left\{\frac{2x(2-x)}{1-x}\right.\\&\left.\quad+x\log[m^2(1-x)]\right\},\\c_2&=-\frac{2x(2-x)}{1-x}-x\log[m^2(1-x)].\end{aligned} \tag{28}$$

The final renormalized expression for $\Sigma_2(p)$ is

$$\begin{aligned}-i\Sigma_2(p)=-\frac{e^2}{8\pi^2}\int_0^1 dx\left\{(m-p\!\!\!/)\left[\frac{2x(2-x)}{1-x}+x\log[m^2(1-x)]\right]\right.\\\left.-(2m-xp\!\!\!/)\log(m^2-xp^2)+m(2-x)\log[m^2(1-x)]\right\},\end{aligned} \tag{29}$$

in agreement with the result obtained by dimensional regularization [3]. The second two conditions in Eq. (23) involve the photon self energy [3]

$$\begin{aligned}i\Pi_2^{\mu\nu}(q^2)=-4ie^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\\\times\frac{g^{\mu\nu}[\frac{1}{2}\ell^2+m^2+x(1-x)q^2]-2x(1-x)q^\mu q^\nu}{(\ell^2+m^2-x(1-x)q^2)^2}.\end{aligned} \tag{30}$$

Differentiating twice with respect to q^2 and using the relation $q^\mu q^\nu=\frac{1}{4}g^{\mu\nu}q^2$,

$$\begin{aligned}i\frac{d^2\Pi_2^{\mu\nu}(q)}{dq^4}=-i\frac{e^2}{2\pi^2}\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}g^{\mu\nu}[x(1-x)]^2\\\times\left[\frac{1}{m^2-x(1-x)q^2}+\frac{1}{2}\frac{(m^2+\frac{1}{2}x(1-x)q^2)}{[m^2-x(1-x)q^2]^2}\right].\end{aligned} \tag{31}$$

Integrating twice with respect to q^2,

$$i\Pi_2^{\mu\nu}(q)=-i\frac{e^2}{2\pi^2}\int_0^1 dx\frac{3}{4}g^{\mu\nu}\Big[m^2\log(-1)+c_1+c_2q^2 -x(1-x)q^2\log[m^2-x(1-x)q^2]\Big]. \quad (32)$$

Imposing the renormalization conditions in Eq. (23),

$$c_2=x(1-x)\log(m^2), \quad c_1=-m^2\log(-1). \quad (33)$$

Thus, the final expression for the photon self energy is

$$i\Pi_2^{\mu\nu}(q)=i\frac{e^2}{2\pi^2}(q^2g^{\mu\nu}-q^\mu q^\nu) \times\int_0^1 dx x(1-x)\log\left[\frac{m^2-x(1-x)q^2}{m^2}\right], \quad (34)$$

consistent with the Ward identity and in agreement with the standard result [2, 3]. The last condition involves the electron vertex function

$$\delta\Gamma^\mu(\Delta)=2ie^2\int_0^1 dxdydz\delta(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\frac{2}{(\ell^2-\Delta)^3} \times\Bigg(\gamma^\mu\left[\frac{1}{2}\ell^2+(1-x)(1-y)q^2+(1-4z+z^2)m^2\right] +\frac{i\sigma^{\mu\nu}q_\nu}{2m}\left(2m^2z(1-z)\right)\Bigg), \quad (35)$$

where $\Delta=-xyq^2+(1-z)^2m^2$. Differentiating once with respect to Δ, and integrating over ℓ,

$$\frac{d}{d\Delta}\delta\Gamma^\mu(\Delta)=\frac{e^2}{8\pi^2}\int_0^1 dxdydz\delta(x+y+z-1) \times\Bigg(\gamma^\mu\left[-\frac{1}{\Delta}-\frac{1}{\Delta^2}\left((1-x)(1-y)q^2+(1-4z+z^2)m^2\right)\right] -\frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[\frac{1}{\Delta^2}2m^2z(1-z)\right]\Bigg). \quad (36)$$

Integrating with respect to Δ,

$$\delta\Gamma^\mu(\Delta)=\frac{e^2}{8\pi^2}\int_0^1 dxdydz\delta(x+y+z-1) \times\Bigg(\gamma^\mu\left[-\log\Delta+c_1+\frac{1}{\Delta}\left((1-x)(1-y)q^2+(1-4z+z^2)m^2\right)\right] +\frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[\frac{1}{\Delta}2m^2z(1-z)\right]\Bigg). \quad (37)$$

Imposing the initial condition gives

$$c_1=\log[(1-z)^2m^2]-\frac{1-4z+z^2}{(1-z)^2}. \quad (38)$$

Thus, the final renormalized expression for the electron vertex is

$$\delta\Gamma^\mu(\Delta)=\frac{e^2}{8\pi^2}\int_0^1 dxdydz\delta(x+y+z-1) \times\Bigg(\gamma^\mu\left[-\log\Delta+c_1+\frac{1}{\Delta}\left((1-x)(1-y)q^2+(1-4z+z^2)m^2\right)\right] +\frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[\frac{1}{\Delta}2m^2z(1-z)\right]\Bigg), \quad (39)$$

in agreement with the standard result [2, 3]. This completes the one-loop renormalization of QED.

SUMMARY

In the traditional approach of renormalization, divergent loop integrals are regulated by introducing an arbitrary parameter, and then combined with counterterms and renormalization conditions to obtain a renormalized physical result independent of the regulator. While successful, this approach is tedious and must be applied with caution to ensure gauge-invariant results. More importantly, the need for regularization of any form raises questions about the logical foundations of quantum field theory. In this article, I address this issue head on by showing that when renormalization conditions are recast as initial conditions for momentum-space differential equations, the need for regularization disappears because the process of differentiation under the loop integrals renders them finite. I applied this method to successfully renormalize scalar ϕ^4 theory and QED without introducing a regulator or counterterms. Beyond considerable technical simplifications, the ability to perform renormalization without introducing a regulator or counterterms may provide a more fundamental formulation of quantum field theory free of ultraviolet divergences.

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