On Catalan's Constant: Upgrade 1

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abstract
This note presents an integral for Catalan's constant

Introduction

Catalan´s constant , named after Eugéne Charles Catalan (1814-1894) and usually denoted by G , is defined by

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots \tag{1}$$

The numerical value is

$$G=0.915965594177219015054603514932\ldots \tag{2}$$

It is not known whether G is irrational.

Some representations of Catalan's constant:

$$G=\int_0^1\frac{\tan^{-1}x}{x}dx \tag{3}$$

$$G=\frac{1}{2}\int_0^{\infty}\frac{x}{\cosh x}dx \tag{4}$$

$$G=\int_1^{\infty}\frac{\ln x}{1+x^2}dx \tag{5}$$

$$G=\frac{1}{2}\int_0^{\pi/2}\frac{x}{\sin x}dx \tag{6}$$

$$G=\int_0^1\int_0^1\frac{1}{1+x^2y^2}dxdy \tag{7}$$

In this note we give an integral for Catalan's constant.

Integral for Catalan’s constant

Let $\alpha = \tan^{-1}\left(\frac{\sqrt{2\sqrt{3}-3}}{3}\right)$, then

$$\frac{3}{2}G = \int_0^{\alpha} \ln\left(f(x)\right)dx + \int_{\alpha}^{\pi/2} \ln\left(g(x)\right)dx \tag{8}$$

$$f(x) = \frac{\cot x}{3}\left(1 + 2\sqrt{1+3\tan x}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{2+9\tan^2 x + 27\tan^3 x}{2\sqrt{\left(1+3\tan x\right)^3}}\right)\right)\right) \tag{9}$$

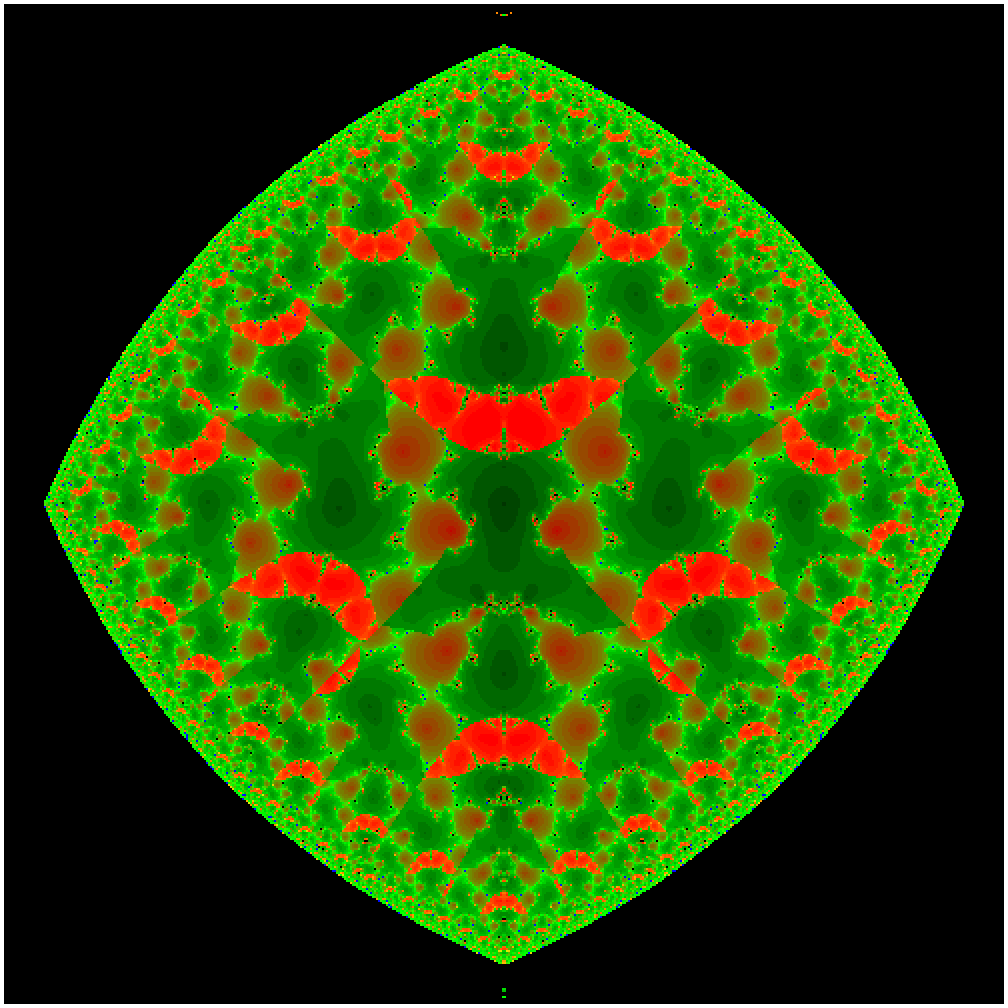
$$\begin{aligned} g(x) = \frac{\cot x}{3} + \frac{1}{3\sqrt[3]{2}}\left(27 + 9\cot^2 x + 2\cot^3 x - 3\sqrt{81 + 54\cot^2 x - 3\cot^4 x}\right)^{1/3} + \\ + \frac{1}{3\sqrt[3]{2}}\left(27 + 9\cot^2 x + 2\cot^3 x + 3\sqrt{81 + 54\cot^2 x - 3\cot^4 x}\right)^{1/3} \end{aligned} \tag{10}$$

- Change of varibles: $u = \tan x$, $dx = \frac{1}{1+u^2}du$.
- $\beta = \frac{\sqrt{2\sqrt{3}-3}}{3}$

$$\frac{3}{2}G = \int_0^{\beta} \frac{\ln F(x)}{1+x^2}dx + \int_{\beta}^{\infty} \frac{\ln G(x)}{1+x^2}dx \tag{11}$$

$$F(x) = \frac{1}{3x}\left(1 + 2\sqrt{1+3x}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{2+9x+27x^3}{2\sqrt{\left(1+3x\right)^3}}\right)\right)\right) \tag{12}$$

$$\begin{aligned} G(x) = \frac{1}{3x} + \frac{1}{3\sqrt[3]{2}}\left(27 + \frac{9}{x^2} + \frac{2}{x^3} - 3\sqrt{81 + \frac{54}{x^2} - \frac{3}{x^4}}\right)^{1/3} + \\ + \frac{1}{3\sqrt[3]{2}}\left(27 + \frac{9}{x^2} + \frac{2}{x^3} + 3\sqrt{81 + \frac{54}{x^2} - \frac{3}{x^4}}\right)^{1/3} \end{aligned} \tag{13}$$



A Fractal Image

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