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$\beta-$Change of Finsler Metric by h-Vector and Imbedding Classes of Their Tangent Spaces

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Abstract: We have considered the $\beta-$change of Finsler metric L given by $\overline{L} = f(L, \beta)$ where f is any positively homogeneous function of degree one in L and β. Here $\beta = b_i(x, y)y^i$, in which b_i are components of a covariant h-vector in Finsler space F^n with metric L. We have obtained that due to this change of Finsler metric, the imbedding class of their tangent Riemannian space is increased at the most by two.

Key Words: $\beta-$Change of Finsler metric, h-vector, imbedding class.

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§1. Introduction

Let (M^n, L) be an n-dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function $L(x, y)$. In 1971, Matsumoto [2] introduced the transformation of Finsler metric given by

$$\overline{L}(x, y) = L(x, y) + \beta(x, y), \tag{1.1}$$

$$\overline{L}^2(x, y) = L^2(x, y) + \beta^2(x, y), \tag{1.2}$$

where $\beta(x, y) = b_i(x)y^i$ is a one-form on M^n. He has proved the following.

Theorem A. *Let $(M^n, \overline{L})$ be a locally Minkowskian n-space obtained from a locally Minkowskian n-space (M^n, L) by the change (1.1). If the tangent Riemannian n-space (M^n_x, g_x) to (M^n, L) is of imbedding class r, then the tangent Riemannian n-space $(M^n_x, \overline{g}_x)$ to $(M^n, \overline{L})$ is of imbedding class at most $r + 2$.*

Theorem B. *Let $(M^n, \overline{L})$ be a locally Minkowskian n-space obtained from a locally Minkowskian n-space (M^n, L) by the change (1.2). If the tangent Riemannian n-space (M^n_x, g_x) to (M^n, L) is of imbedding class r, then the tangent Riemannian n-space $(M^n_x, \overline{g}_x)$ to $(M^n, \overline{L})$ is of imbedding class at most $r + 1$.*

Theorem B is included in theorem A due to the phrase "*at most*".

In [6] Singh, Prasad and Kumari Bindu have proved that the theorem A is valid for Kropina

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change of Finsler metric given by

$$\overline{L}(x,y) = \frac{L^2(x,y)}{\beta(x,y)}.$$

In [3], Prasad, Shukla and Pandey have proved that the theorem A is also valid for exponential change of Finsler metric given by

$$\overline{L}(x,y) = Le^{\beta/L}.$$

Recently Prasad and Kumari Bindu [5] have proved that the theorem A is valid for $\beta-$change [7] given by

$$\overline{L}(x,y) = f(L,\beta),$$

where f is any positively homogeneous function of degree one in L, β and β is one-form.

In all these works it has been assumed that $b_i(x)$ in β is a function of positional coordinate only.

The concept of $h-$vector has been introduced by H.Izumi. The covariant vector field $b_i(x,y)$ is said to be $h-$vector if $\frac{\partial b_i}{\partial y^j}$ is proportional to angular metric tensor.

In 1990, Prasad, Shukla and Singh [4] have proved that the theorem A is valid for the transformation (1.1) in which b_i in β is $h-$vector.

All the above $\beta-$changes of Finsler metric encourage the authors to check whether the theorem A is valid for any change of Finsler metric by $h-$vector.

In this paper we have proved that the theorem A is valid for the $\beta-$change of Finsler metric given by

$$\overline{L}(x,y) = f(L,\beta), \tag{1.3}$$

where f is positively homogeneous function of degree one in L, β and

$$\beta(x,y) = b_i(x,y)y^i. \tag{1.4}$$

Here $b_i(x,y)$ are components of a covariant $h-$vector satisfying

$$\frac{\partial b_i}{\partial y^j} = \rho h_{ij}, \tag{1.5}$$

where ρ is any scalar function of x, y and h_{ij} are components of angular metric tensor. The homogeneity of f gives

$$Lf_1 + \beta f_2 = f, \tag{1.6}$$

where the subscripts 1 and 2 denote the partial derivatives with respect to L and β respectively.

Differentiating (1.6) with respect to L and β respectively, we get

$$Lf_{11} + \beta f_{12} = 0 \quad \text{and} \quad Lf_{12} + \beta f_{22} = 0.$$

Hence, we have
$$\frac{f_{11}}{\beta^2} = -\frac{f_{12}}{\beta L} = \frac{f_{22}}{L^2}$$
which gives
$$f_{11} = \beta^2\omega, \quad f_{22} = L^2\omega, \quad f_{12} = -\beta L\omega, \tag{1.7}$$
where Weierstrass function ω is positively homogeneous function of degree -3 in L and β. Therefore
$$L\omega_1 + \beta\omega_2 + 3\omega = 0. \tag{1.8}$$
Again ω_1 and ω_2 are positively homogeneous function of degree - 4 in L and β, so
$$L\omega_{11} + \beta\omega_{12} + 4\omega_1 = 0 \quad \text{and} \quad L\omega_{21} + \beta\omega_{22} + 4\omega_2 = 0. \tag{1.9}$$

Throughout the paper we frequently use equation (1.6) to (1.9) without quoting them.

§2. An $h-$Vector

Let $b_i(x, y)$ be components of a covariant vector in the Finsler space (M^n, L). It is called an $h-$vector if there exists a scalar function ρ such that
$$\frac{\partial b_i}{\partial y^j} = \rho h_{ij}, \tag{2.1}$$
where h_{ij} are components of angular metric tensor given by
$$h_{ij} = g_{ij} - l_i l_j = L\frac{\partial^2 L}{\partial y^i \partial y^j}.$$

Differentiating (2.1) with respect to y^k, we get
$$\dot\partial_j\dot\partial_k b_i = (\dot\partial_k\rho)h_{ij} + \rho L^{-1}\{L^2\dot\partial_i\dot\partial_j\dot\partial_k L + h_{ij}l_k\},$$
where $\dot\partial_i$ stands for $\frac{\partial}{\partial y^i}$.

The skew-symmetric part of the above equation in j and k gives
$$(\dot\partial_k\rho + \rho L^{-1}l_k)h_{ij} - (\dot\partial_j\rho + \rho L^{-1}l_j)h_{ik} = 0.$$
Contracting this equation by g^{ij}, we get
$$(n-2)[\dot\partial_k\rho + \rho L^{-1}l_k] = 0,$$
which for $n > 2$, gives
$$\dot\partial_k\rho = -\frac{\rho}{L}l_k, \tag{2.2}$$
where we have used the fact that ρ is positively homogeneous function of degree -1 in y^i, i.e.,

$\frac{\partial\rho}{\partial y^j}y^j = -\rho.$

We shall frequently use equation (2.2) without quoting it in the next article.

§3. Fundamental Quantities of $(M^n, \overline{L})$

To find the relation between fundamental quantities of (M^n, L) and $(M^n, \overline{L})$, we use the following results

$$\dot{\partial}_i\beta = b_i, \quad \dot{\partial}_i L = l_i, \quad \dot{\partial}_j l_i = L^{-1}h_{ij}. \tag{3.1}$$

The successive differentiation of (1.3) with respect to y^i and y^j give

$$\overline{l}_i = f_1 l_i + f_2 b_i, \tag{3.2}$$

$$\overline{h}_{ij} = \frac{fp}{L}h_{ij} + fL^2 w m_i m_j, \tag{3.3}$$

where

$$p = f_1 + Lf_2\rho, \quad m_i = b_i - \frac{\beta}{L}l_i.$$

The quantities corresponding to $(M^n, \overline{L})$ will be denoted by putting bar on the top of those quantities.

From (3.2) and (3.3) we get the following relations between metric tensors of (M^n, L) and $(M^n, \overline{L})$

$$\begin{aligned} \overline{g}_{ij} &= \frac{fp}{L}g_{ij} - L^{-1}\{\beta(f_1 f_2 - f\beta L\omega) + L\rho f f_2\}l_i l_j \\ &\quad + (fL^2\omega + f_2^2)b_i b_j + (f_1 f_2 - f\beta L\omega)(l_i b_j + l_j b_i). \end{aligned} \tag{3.4}$$

The contravariant components of the metric tensor of $(M^n, \overline{L})$ will be obtained from (3.4) as follows:

$$\overline{g}^{ij} = \frac{L}{fp}g^{ij} + \frac{Lv}{f^3 pt}l^i l^j - \frac{L^4\omega}{fpt}b^i b^j - \frac{L^2 u}{f^2 pt}(l^i b^j + l^j b^i), \tag{3.5}$$

where, we put $b^i = g^{ij}b_j,\ l^i = g^{ij}l_j,\ b^2 = g^{ij}b_i b_j$ and

$$\begin{aligned} u &= f_1 f_2 - f\beta L\omega + L\rho f_2^2, \\ v &= (f_1 f_2 - f\beta L\omega)(f\beta + \triangle f_2 L^2) + L\rho f_2\{f(f + L^2\rho f_2) \\ &\quad + L^2\triangle(f_2^2 + fL^2\omega)\} \end{aligned}$$

and

$$t = f_1 + L^3\omega\triangle + Lf_2\rho, \qquad \triangle = b^2 - \frac{\beta^2}{L^2}. \tag{3.6}$$

Putting $q = f_1 f_2 - f\beta L\omega + L\rho(f_2^2 + fL^2\omega)$, $s = 3f_2\omega + f\omega_2$, we find that

$$\begin{array}{ll}
(a) & \dot{\partial}_i f = \frac{f}{L} l_i + f_2 m_i \\
(b) & \dot{\partial}_i f_1 = -\beta L\omega m_i \\
(c) & \dot{\partial}_i f_2 = L^2\omega m_i \\
(d) & \dot{\partial}_i p = -L\omega(\beta - \rho L^2) m_i \\
(e) & \dot{\partial}_i \omega = -\frac{3\omega}{L} l_i + \omega_2 m_i \\
(f) & \dot{\partial}_i b^2 = -2C_{..i} + 2\rho m_i \\
(g) & \dot{\partial}_i \triangle = -2C_{..i} - \frac{2}{L^2}(\beta - \rho L^2) m_i,
\end{array} \tag{3.7}$$

$$\begin{array}{ll}
(a) & \dot{\partial}_i q = -(\beta - \rho L^2) s L m_i \\
(b) & \dot{\partial}_i t = -2L^3\omega C_{..i} + [L^3\triangle\omega_2 - 3(\beta - \rho L^2)L\omega] m_i \\
(c) & \dot{\partial}_i s = -\frac{3s}{L} l_i + (4f_2\omega_2 + 3\omega^2 L^2 + f\omega_{22}) m_i
\end{array} \tag{3.8}$$

where "." denotes the contraction with b^i, viz. $C_{..i} = C_{jki} b^j b^k$.

Differentiating (3.4) with respect to y^k and using (d that

$$m_i l^i = 0, \quad m_i m^i = \triangle = m_i b^i, \quad h_{ij} m^j = h_{ij} b^j = m_i, \tag{3.10}$$

where $m^i = g^{ij} m_j = b^i - \frac{\beta}{L} l^i$.

To find $\overline{C}^i_{jk} = \overline{g}^{ih}\overline{C}_{jhk}$ we use (3.5), (3.9), (3.10) and get

$$\begin{aligned}
\overline{C}^i_{jk} &= C^i_{jk} + \frac{q}{2fp}(h_{jk} m^i + h^i_j m_k + h^i_k m_j) + \frac{sL^3}{2fp} m_j m_k m^i - \frac{L}{ft} C_{.jk} n^i \\
&\quad - \frac{Lq\triangle}{2f^2pt} h_{jk} n^i - \frac{2Lq + L^4\triangle s}{2f^2pt} m_j m_k n^i,
\end{aligned} \tag{3.11}$$

where $n^i = fL^2\omega b^i + u l^i$.

Corresponding to the vectors with components n^i and m^i, we have the following:

$$C_{ijk} m^i = C_{.jk}, \quad C_{ijk} n^i = fL^2\omega C_{.jk}, \quad m_i n^i = fL^2\omega\triangle. \tag{3.12}$$

To find the v-curvature tensor of $(M^n, \overline{L})$ with respect to Cartan's connection, we use the following:

$$C^h_{ij} h_{hk} = C_{ijk}, \quad h^i_k h^k_j = h^i_j, \quad h_{ij} n^i = fL^2\omega m_j. \tag{3.13}$$

The v-curvature tensors $\overline{S}_{hijk}$ of $(M^n, \overline{L})$ is defined as

$$\overline{S}_{hijk} = \overline{C}^r_{hk} C_{hjr} - \overline{C}^r_{hj}\overline{C}_{ikr}. \tag{3.14}$$

From (3.9)–(3.14), we get the following relation between v-curvature tensors of (M^n, L)

and $(M^n, \overline{L})$:

$$\overline{S}_{hijk} = \frac{fp}{L} S_{hijk} + d_{hj}d_{ik} - d_{hk}d_{ij} + E_{hk}E_{ij} - E_{hj}E_{ik}, \tag{3.15}$$

where

$$d_{ij} = PC_{.ij} - Qh_{ij} + Rm_i m_j, \tag{3.16}$$

$$E_{ij} = Sh_{ij} + Tm_i m_j, \tag{3.17}$$

$$P = L\left(\frac{fp\omega}{t}\right)^{1/2}, \quad Q = \frac{pq}{2L^2\sqrt{fp\omega t}}, \quad R = \frac{L(2\omega q - sp)}{2\sqrt{f\omega pt}},$$

$$S = \frac{q}{2L^2\sqrt{f\omega}}, \quad T = \frac{L(sp - \omega q)}{2p\sqrt{f\omega}}.$$

§4. **Imbedding Class Numbers**

The tangent vector space M_x^n to M^n at every point x is considered as the Riemannian n-space (M_x^n, g_x) with the Riemannian metric $g_x = g_{ij}(x,y)dy^i dy^j$. Then the components of the Cartan's tensor are the Christoffel symbols associated with g_x:

$$C^i_{jk} = \frac{1}{2} g^{ih}(\dot{\partial}_k g_{jh} + \dot{\partial}_j g_{hk} - \dot{\partial}_h g_{jk}).$$

Thus C^i_{jk} defines the components of the Riemannian connection on M_x^n and v-covariant derivative, say

$$X_i|_j = \dot{\partial}_j X_i - X_h C^h_{ij}$$

is the covariant derivative of covariant vector X_i with respect to Riemannian connection C^i_{jk} on M_x^n. It is observed that the v-curvature tensor S_{hijk} of (M^n, L) is the Riemannian Christoffel curvature tensor of the Riemannian space (M^n, g_x) at a point x. The space (M^n, g_x) equipped with such a Riemannian connection is called the tangent Riemannian n-space [2].

It is well known [1] that any Riemannian n-space V^n can be imbedded isometrically in a Euclidean space of dimension $\frac{n(n+1)}{2}$. If $n+r$ is the lowest dimension of the Euclidean space in which V^n is imbedded isometrically, then the integer r is called the imbedding class number of V^n. The fundamental theorem of isometric imbedding ([1] page 190) is that the tangent Riemannian n-space (M_x^n, g_x) is locally imbedded isometrically in a Euclidean $(n+r)-$space if and only if there exist $r-$number $\epsilon_P = \pm 1$, $r-$symmetric tensors $H_{(P)ij}$ and $\frac{r(r-1)}{2}$ covariant vector fields $H_{(P,Q)i} = -H_{(Q,P)i}$; $P, Q = 1, 2, \cdots, r$, satisfying the Gauss equations

$$S_{hijk} = \sum_P \epsilon_P \{H_{(P)hj}H_{(P)ik} - H_{(P)ij}H_{(P)hk}\}, \tag{4.1}$$

The Codazzi equations

$$H_{(P)ij}|_k - H_{(P)ik}|_j = \sum_Q \epsilon_Q \{H_{(Q)ij}H_{(Q,P)k} - H_{(Q)ik}H_{(Q,P)j}\}, \tag{4.2}$$

and the Ricci-Kühne equations

$$\begin{aligned} H_{(P,Q)i}|_j - H_{(P,Q)j}|_i &+ \sum_R \epsilon_R\{H_{(R,P)i}H_{(R,Q)j} - H_{(R,P)j}H_{(R,Q)i}\} \\ &+ g^{hk}\{H_{(P)hi}H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\} = 0. \end{aligned} \tag{4.3}$$

The numbers $\epsilon_P = \pm 1$ are the indicators of unit normal vector N_P to M^n and $H_{(P)ij}$ are the second fundamental tensors of M^n with respect to the normals N_P. Thus if g_x is assumed to be positive definite, there exists a Cartesian coordinate system (z^i, u^p) of the enveloping Euclidean space E^{n+r} such that ds^2 in E^{n+r} is expressed as

$$ds^2 = \sum_i (dz^i)^2 + \sum_p \epsilon_p (du^p)^2.$$

§5. Proof of Theorem A

In order to prove the theorem A, we put

$$\begin{aligned} &\text{(a)} \quad \overline{H}_{(P)ij} = \sqrt{\frac{fp}{L}}\, H_{(P)ij}, \qquad \overline{\epsilon}_P = \epsilon_P, \quad P = 1, 2, \cdots, r \\ &\text{(b)} \quad \overline{H}_{(r+1)ij} = d_{ij}, \qquad \overline{\epsilon}_{r+1} = 1 \\ &\text{(c)} \quad \overline{H}_{(r+2)ij} = E_{ij}, \qquad \overline{\epsilon}_{r+2} = -1. \end{aligned} \tag{5.1}$$

Then it follows from (3.15) and (4.1) that

$$\overline{S}_{hijk} = \sum_{\lambda=1}^{r+2} \overline{\epsilon}_\lambda \{\overline{H}_{(\lambda)hj}\overline{H}_{(\lambda)ik} - \overline{H}_{(\lambda)hk}\overline{H}_{(\lambda)ij}\},$$

which is the Gauss equation of $(M^n_x, \overline{g}_x)$.

Moreover, to verify Codazzi and Ricci Kühne equation of $(M^n_x, \overline{g}_x)$, we put

$$\begin{aligned} &\text{(a)} \quad \overline{H}_{(P,Q)i} = -\overline{H}_{(Q,P)i} = H_{(P,Q)i}, \quad P, Q = 1, 2, , \cdots, r \\ &\text{(b)} \quad \overline{H}_{(P,r+1)i} = -\overline{H}_{(r+1,P)i} = \frac{L\sqrt{L\omega}}{\sqrt{t}} H_{(P).i}, \quad P = 1, 2, \cdots, r \\ &\text{(c)} \quad \overline{H}_{(P,r+2)i} = -\overline{H}_{(r+2,P)i} = 0, \quad P = 1, 2, \cdots, r. \\ &\text{(d)} \quad \overline{H}_{(r+1,r+2)i} = -\overline{H}_{(r+2,r+1)i} = \frac{sp - 2q\omega}{2f\omega\sqrt{pt}} m_i. \end{aligned} \tag{5.2}$$

The Codazzi equations of $(M^n_x, \overline{g}_x)$ consists of the following three equations:

$$\begin{aligned} \text{(a)} \quad \overline{H}_{(P)ij}\|_k - \overline{H}_{(P)ik}\|_j = &\sum_Q \overline{\epsilon}_Q \{\overline{H}_{(Q)ij}\overline{H}_{(Q,P)k} - \overline{H}_{(Q)ik}\overline{H}_{(Q,P)j}\} \\ &+ \overline{\epsilon}_{r+1}\{\overline{H}_{(r+1)ij}\overline{H}_{(r+1,P)k} - \overline{H}_{(r+1)ik}\overline{H}_{(r+1,P)j}\} \\ &+ \overline{\epsilon}_{r+2}\{\overline{H}_{(r+2)ij}\overline{H}_{(r+2,P)k} - \overline{H}_{(r+2)ik}\overline{H}_{(r+2,P)k}\} \end{aligned} \tag{5.3}$$

$$\text{(b)}\quad \overline{H}_{(r+1)ij}\|_k - \overline{H}_{(r+1)ik}\|_j = \sum_Q \overline{\epsilon}_Q\{\overline{H}_{(Q)ij}\overline{H}_{(Q,r+1)k} - \overline{H}_{(Q)ik}\overline{H}_{(Q,r+1)j}\} + \overline{\epsilon}_{r+2}\{\overline{H}_{(r+2)ij}\overline{H}_{(r+2,r+1)k} - \overline{H}_{(r+2)ik}\overline{H}_{(r+2,r+1)j}\},$$

$$\text{(c)}\quad \overline{H}_{(r+2)ij}\|_k - \overline{H}_{(r+2)ik}\|_j = \sum_Q \overline{\epsilon}_Q\{\overline{H}_{(Q)ij}\overline{H}_{(Q,r+2)k} - \overline{H}_{(Q)ik}\overline{H}_{(Q,r+2)j}\} + \overline{\epsilon}_{r+1}\{\overline{H}_{(r+1)ij}\overline{H}_{(r+1,r+2)k} - \overline{H}_{(r+1)ik}\overline{H}_{(r+1,r+2)j}\}.$$

where $\|_i$ denotes v-covariant derivative in $(M^n, \overline{L})$, i.e. covariant derivative in tangent Riemannian n-space $(M^n_x, \overline{g}_x)$ with respect to its Christoffel symbols $\overline{C}^i_{jk}$. Thus

$$X_i\|_j = \dot{\partial}_j X_i - X_h \overline{C}^h_{ij}.$$

To prove these equations we note that for any symmetric tensor X_{ij} satisfying $X_{ij}l^i = X_{ij}l^j = 0$, we have from (3.11),

$$\begin{aligned} X_{ij}\|_k - X_{ik}\|_j &= X_{ij}|_k - X_{ik}|_j - \frac{q}{2ft}(h_{ik}X_{.j} - h_{ij}X_{.k}) \\ &+ \frac{L^3\omega}{t}(C_{.ik}X_{.j} - C_{.ij}X_{.k}) - \frac{q}{2fp}(X_{ij}m_k - X_{ik}m_j) \\ &+ \frac{L^3(2q\omega - sp)}{2fpt}(X_{.j}m_k - X_{.k}m_j)\, m_i. \end{aligned} \tag{5.4}$$

Also if X is any scalar function, then $X\|_j = X|_j = \dot{\partial}_j X$.

Verification of (5.3)(a) In view of (5.1) and (5.2), equation (5.3)a is equivalent to

$$\begin{aligned} &\left(\sqrt{\frac{fp}{L}}H_{(P)ij}\right)\Big\|_k - \left(\sqrt{\frac{fp}{L}}H_{(P)ik}\right)\Big\|_j \\ &= \sqrt{\frac{fp}{L}}.\sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q,P)k} - H_{(Q)ik}H_{(Q,P)j}\} - \frac{L\sqrt{L\omega}}{\sqrt{t}}\{d_{ij}H_{(P).k} - d_{ik}H_{(P).j}\}. \end{aligned} \tag{5.5}$$

Since $\left(\sqrt{\frac{fp}{L}}\right)\Big\|_k = \dot{\partial}_k\left(\sqrt{\frac{fp}{L}}\right) = \frac{q}{2\sqrt{fLp}}\, m_k$, applying formula (5.4) for $H_{(P)ij}$, we get

$$\begin{aligned} &\left(\sqrt{\frac{fp}{L}}H_{(P)ij}\right)\Big\|_k - \left(\sqrt{\frac{fp}{L}}H_{(P)ik}\right)\Big\|_j = \sqrt{\frac{fp}{L}}\{H_{(P)ij}|_k - H_{(P)ik}|_j\} \\ &- \frac{q}{2ft}\sqrt{\frac{fp}{L}}\{h_{ik}H_{(P).j} - h_{ij}H_{(P).k}\} + \frac{L^3\omega}{t}\sqrt{\frac{fp}{L}}\{C_{.ik}H_{(p).j} - C_{.ij}H_{(p).k}\} \\ &+ \frac{L^2\sqrt{L}(2q\omega - sp)}{2t\sqrt{fp}}\{H_{(P).j}m_k - H_{(P).k}m_j\}m_i. \end{aligned} \tag{5.6}$$

Substituting the values of $\left(\sqrt{\frac{fp}{L}}H_{(P)ij}\right)\Big\|_k - \left(\sqrt{\frac{fp}{L}}H_{(P)ik}\right)\Big\|_j$ from (5.6) and the values of d_{ij} from (3.16) in (5.5) we find that equation (5.5) is identically satisfied due to equation

(4.2).

Verification of (5.3)(b) In view of (5.1) and (5.2), equation (5.3)b is equivalent to

$$d_{ij}\|_k - d_{ik}\|_j = L\sqrt{\frac{f\omega p}{t}}\sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q).k} - H_{(Q)ik}H_{(Q).j}\} + \frac{sp-2q\omega}{2f\omega\sqrt{pt}}\{E_{ij}m_k - E_{ik}m_j\}. \tag{5.7}$$

To verify (5.7), we note that

$$C_{.ij}|_k - C_{.ik}|_j = -b^h S_{hijk} \tag{5.8}$$

$$h_{ij}|_k - h_{ik}|_j = L^{-1}(h_{ij}l_k - h_{ik}l_j), \tag{5.9}$$

$$m_i|_k = -C_{.ik} - \left(\frac{\beta}{L^2} - \rho\right) h_{ik} - \frac{1}{L} l_i m_k. \tag{5.10}$$

$$\dot{\partial}_k(f\omega p) = -2L^{-1}f\omega p l_k + (q\omega + fp\omega_2)m_k. \tag{5.11}$$

Contracting (3.16) with b^i and using (3.10), we find that

$$d_{.j} = L\sqrt{\frac{f\omega p}{t}}C_{..j} + \frac{q(2L^3\omega\triangle - p) - L^3\triangle sp}{2L^2\sqrt{f\omega pt}}m_j. \tag{5.12}$$

Applying formula (5.4) for d_{ij} and substituting the values of $d_{.j}$ from (5.12) and d_{ij} from (3.16), we get

$$\begin{aligned} d_{ij}\|_k - d_{ik}\|_j = \; & d_{ij}|_k - d_{ik}|_j - \frac{Lq\sqrt{f\omega p}}{2ft^{3/2}}(h_{ik}C_{..j} - h_{ij}C_{..k}) \\ & + \frac{L^4\omega(2q\omega - sp)}{2\sqrt{f\omega p}.t^{3/2}}(C_{..j}m_k - C_{..k}m_j)m_i \\ & + \frac{L^4\omega\sqrt{f\omega p}}{t^{3/2}}(C_{.ik}C_{..j} - C_{.ij}C_{..k}) \\ & + \frac{L^4\omega\triangle(3q\omega - sp)}{2\sqrt{f\omega p}.t^{3/2}}(C_{.ik}m_j - C_{.ij}m_k) \\ & - \frac{Lq\triangle(3q\omega - sp)}{4f\sqrt{f\omega p}.t^{3/2}}(h_{ik}m_j - h_{ij}m_k). \end{aligned} \tag{5.13}$$

From (3.16), we obtain

$$\begin{aligned} d_{ij}|_k - d_{ik}|_j \; & = P(C_{.ij}|_k - C_{.ik}|_j) - Q(h_{ij}|_k - h_{ik}|_j) \\ & + R(m_i|_k m_j + m_j|_k m_i - m_i|_j m_k - m_k|_j m_i) \\ & + (\dot{\partial}_k P)C_{.ij} - (\dot{\partial}_j P)C_{.ik} - (\dot{\partial}_k Q)h_{ij} + (\dot{\partial}_j Q)h_{ik}) \\ & + (\dot{\partial}_k R)m_i m_j - (\dot{\partial}_j R)m_i m_k). \end{aligned} \tag{5.14}$$

Since,

$$\begin{aligned}\dot\partial_k P &= \frac{L^4\omega\sqrt{f\omega p}}{t^{3/2}}C_{..k} + \left[\frac{Lfp\{p\omega_2 + 3L\omega^2(\beta-\rho L^2)\}}{2\sqrt{f\omega p}.t^{3/2}}\right.\\ &\left.+\frac{Lq\omega}{2\sqrt{f\omega pt}}\right]m_k,\\ \dot\partial_k Q &= \frac{Lpq\omega}{2\sqrt{f\omega p}.t^{3/2}}C_{..k} - \frac{pq}{2L^3\sqrt{f\omega pt}}l_k\\ &-\frac{(\beta-\rho L^2)(q\omega+sp)}{2L\sqrt{f\omega pt}}m_k - \frac{pq(q\omega+fp\omega_2)}{4L^2(f\omega p)^{3/2}\sqrt{t}}m_k\\ &+\frac{pq\{3\omega(\beta-\rho L^2)-L^2\triangle\omega_2\}}{4L\sqrt{f\omega p}\,t^{3/2}}m_k \end{aligned}\tag{5.15}$$

and

$$\dot\partial_k R = \frac{L^4\omega(2q\omega-sp)}{2\sqrt{f\omega p}.t^{3/2}}C_{..k} - \frac{2q\omega-sp}{2\sqrt{f\omega pt}}l_k + \text{term containing } m_k,$$

where we have used the equations (3.6), (3.7) and (3.8).

From equations (5.8)–(5.15), we have

$$\begin{aligned} d_{ij}|_k - d_{ik}|_j &= L\sqrt{\frac{f\omega p}{t}}(-b^h S_{hijk})\\ &+\frac{L^4\omega\triangle(3q\omega-sp)}{2\sqrt{f\omega p}.t^{3/2}}(C_{.ij}m_k - C_{.ik}m_j)\\ &+\frac{L^4\omega\sqrt{f\omega p}}{t^{3/2}}(C_{.ij}C_{..k}-C_{.ik}C_{..j})\\ &+\frac{L\omega pq}{2\sqrt{f\omega p}.t^{3/2}}(h_{ik}C_{..j}-h_{ij}C_{..k})\\ &+\frac{pq[q\omega t + f(L^3\omega\triangle + t)\{3L\omega^2(\beta-\rho L^2)+p\omega_2\}]}{4L^2(f\omega pt)^{3/2}}\times\\ &(h_{ij}m_k - h_{ik}m_j) + \frac{L^4\omega(2q\omega-sp)}{2\sqrt{f\omega p}.t^{3/2}}(C_{..k}m_j - C_{..j}m_k)\,m_i. \end{aligned}\tag{5.16}$$

Substituting the value of $d_{ij}|_k - d_{ik}|_j$ from (5.16) in (5.13), then value of $d_{ij}\|_k - d_{ik}\|_j$ thus obtained in (5.7), and using equations (4.1) and (3.17), it follows that equation (5.7) holds identically.

Verification of (5.3)(c) In view of (5.1) and (5.2), equation (5.3)c is equivalent to

$$E_{ij}\|_k - E_{ik}\|_j = \frac{sp-2q\omega}{2f\omega\sqrt{pt}}(d_{ij}m_k - d_{ik}m_j).\tag{5.17}$$

Contracting (3.17) by b^i and using equation (3.10), we find that

$$E_{.j} = \frac{pq + L^3\triangle(sp-q\omega)}{2L^2p\sqrt{f\omega}}\,m_j.\tag{5.18}$$

Applying formula (5.4) for E_{ij} and substituting the value of $E_{.j}$ from (5.18) and the value of E_{ij} from (3.17), we get

$$\begin{aligned} E_{ij}\|_k - E_{ik}\|_j &= E_{ij}|_k - E_{ik}|_j + \frac{qL\triangle(sp-2q\omega)}{4fpt\sqrt{f\omega}}(h_{ij}m_k - h_{ik}m_j) \\ &+ \frac{L\omega\{pq + L^3\triangle(sp-q\omega)\}}{2pt\sqrt{f\omega}}(C_{.ik}m_j - C_{.ij}m_k). \end{aligned} \tag{5.19}$$

From (3.17), we get

$$\begin{aligned} E_{ij}|_k - E_{ik}|_j &= S(h_{ij}|_k - h_{ik}|_j) + T\{m_i|_k m_j + m_j|_k m_i \\ &-m_i|_j m_k - m_k|_j m_i\} + (\dot\partial_k S)h_{ij} \\ &-(\dot\partial_j S)h_{ik} + (\dot\partial_k T)m_i m_j - (\dot\partial_j T)m_i m_k. \end{aligned} \tag{5.20}$$

Now,

$$(\dot\partial_k S) = -\frac{q}{2L^3\sqrt{f\omega}}l_k - \left[\frac{(\beta-\rho L^2)s}{2L\sqrt{f\omega}} + \frac{q(f\omega_2 + f_2\omega)}{4L^2(f\omega)^{3/2}}\right]m_k \tag{5.21}$$

and

$$(\dot\partial_k T) = -\frac{sp-q\omega}{2p\sqrt{f\omega}}l_k + \text{term containing } m_k,$$

where we have used the equations (3.7) and (3.8).

From equation (5.9)–(5.11), (5.20) and (5.21), we get

$$\begin{aligned} E_{ij}|_k - E_{ik}|_j &= \frac{L(sp-q\omega)}{2p\sqrt{f\omega}}(C_{.ij}m_k - C_{.ik}m_j) \\ &-\frac{q(sp-2q\omega)}{4L^2p(f\omega)^{3/2}}(h_{ij}m_k - h_{ik}m_j). \end{aligned} \tag{5.22}$$

Substituting the value of $E_{ij}|_k - E_{ik}|_j$ from (5.22) in (5.19), then the value of $E_{ij}\|_k - E_{ik}\|_j$ thus obtained in (5.17), and then using (3.16) in the right-hand side of (5.17), we find that the equation (5.17) holds identically.

This completes the proof of Codazzi equations of $(M^n_x, \overline{g}_x)$. The Ricci Kühne equations of $(M^n_x, \overline{g}_x)$ consist of the following four equations

$$\begin{aligned} \text{(a)} \quad & \overline{H}_{(P,Q)i}\|_j - \overline{H}_{(P,Q)j}\|_i + \sum_R \overline{\epsilon}_R\{\overline{H}_{(R,P)i}\overline{H}_{(R,Q)j} \\ &-\overline{H}_{(R,P)j}\overline{H}_{(R,Q)i}\} + \overline{\epsilon}_{r+1}\{\overline{H}_{(r+1,P)i}\overline{H}_{(r+1,Q)j} \\ &-\overline{H}_{(r+1,P)j}\overline{H}_{(r+1,Q)i}\} + \overline{\epsilon}_{r+2}\{\overline{H}_{(r+2,P)i}\overline{H}_{(r+2,Q)j} \\ &-\overline{H}_{(r+2,P)j}\overline{H}_{(r+2,Q)i}\} + \overline{g}^{hk}\{\overline{H}_{(P)hi}\overline{H}_{(Q)kj} \\ &-\overline{H}_{(P)hj}\overline{H}_{(Q)ki}\} = 0, \qquad P,Q = 1,2,\cdots,r \end{aligned} \tag{5.23}$$

$$\begin{aligned}
\text{(b)}\quad & \overline{H}_{(P,r+1)i}\|_j - \overline{H}_{(P,r+1)j}\|_i + \sum_R \overline{\epsilon}_R\{\overline{H}_{(R,P)i}\overline{H}_{(R,r+1)j} - \overline{H}_{(R,P)j}\overline{H}_{(R,r+1)i}\} \\
& + \overline{\epsilon}_{r+2}\{\overline{H}_{(r+2,P)i}\overline{H}_{(r+2,r+1)j} - \overline{H}_{(r+2,P)j}\overline{H}_{(r+2,r+1)i}\} \\
& + \overline{g}^{hk}\{\overline{H}_{(P)hi}\overline{H}_{(r+1)kj} - \overline{H}_{(P)hj}\overline{H}_{(r+1)ki}\} = 0, \quad P = 1,2,\cdots,r \\
\text{(c)}\quad & \overline{H}_{(P,r+2)i}\|_j - \overline{H}_{(P,r+2)j}\|_i + \sum_R \overline{\epsilon}_R\{\overline{H}_{(R,P)i}\overline{H}_{(R,r+2)j} - \overline{H}_{(R,P)j}\overline{H}_{(R,r+2)i}\} \\
& + \overline{\epsilon}_{r+1}\{\overline{H}_{(r+1,P)i}\overline{H}_{(r+1,r+2)j} - \overline{H}_{(r+1,P)j}\overline{H}_{(r+1,r+2)i}\} \\
& + \overline{g}^{hk}\{\overline{H}_{(P)hi}\overline{H}_{(r+2)kj} - \overline{H}_{(P)hj}\overline{H}_{(r+2)ki}\} = 0, \quad P = 1,2,\cdots,r
\end{aligned}$$

$$\begin{aligned}
\text{(d)}\quad & \overline{H}_{(r+1,r+2)i}\|_j - \overline{H}_{(r+1,r+2)j}\|_i + \sum_R \overline{\epsilon}_R\{\overline{H}_{(R,r+1)i}\overline{H}_{(R,r+2)j} - \overline{H}_{(R,r+1)j} \\
& \times\overline{H}_{(R,r+2)i}\} + \overline{g}^{hk}\{\overline{H}_{(r+1)hi}\overline{H}_{(r+2)kj} - \overline{H}_{(r+1)hj}\overline{H}_{(r+2)ki}\} = 0.
\end{aligned}$$

Verification of (5.23)(a) In view of (5.1) and (5.2), equation (5.23)a is equivalent to

$$\begin{aligned}
& H_{(P,Q)i}\|_j - H_{(P,Q)j}\|_i + \sum_R \epsilon_R\{H_{(R,P)i}H_{(R,Q)j} - H_{(R,P)j}H_{(R,Q)i}\} \\
& + \frac{L^3\omega}{t}\{H_{(P).i}H_{(Q).j} - H_{(P).j}H_{(Q).i}\} + \overline{g}^{hk}\{H_{(P)hi}H_{(Q)kj} \\
& - H_{(P)hj}H_{(Q)ki}\}\frac{fp}{L} = 0. \qquad P,Q = 1,2,\ldots,r. \qquad (5.24)
\end{aligned}$$

Since $H_{(P)ij}l^i = 0 = H_{(P)ji}l^i$, from (3.5), we get

$$\begin{aligned}
& \overline{g}^{hk}\{H_{(P)hi}H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\}\frac{fp}{L} = g^{hk}\{H_{(P)hi}H_{(Q)kj} \\
& - H_{(P)hj}H_{(Q)ki}\} - \frac{L^3\omega}{t}\{H_{(P).i}H_{(Q).j} - H_{(P).j}H_{(Q).i}\}.
\end{aligned}$$

Also, we have $H_{(P,Q)i}\|_j - H_{(P,Q)j}\|_i = H_{(P,Q)i}|_j - H_{(P,Q)j}|_i$. Hence equation (5.24) is satisfied identically by virtue of (4.3).

Verification of (5.23)(b) In view of (5.1) and (5.2), equation (5.23)b is equivalent to

$$\begin{aligned}
& \left(\frac{L\sqrt{L\omega}}{\sqrt{t}}H_{(P).i}\right)\Big\|_j - \left(\frac{L\sqrt{L\omega}}{\sqrt{t}}H_{(P).j}\right)\Big\|_i \\
& + \frac{L\sqrt{L\omega}}{\sqrt{t}}\sum_R \epsilon_R\{H_{(R,P)i}H_{(R).j} - H_{(R,P)j}H_{(R).i}\} \\
& + \overline{g}^{hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{\frac{fp}{L}} = 0. \quad P,Q = 1,2,\cdots,r. \qquad (5.25)
\end{aligned}$$

Since $b^h|_j = g^{hk}C_{.jk}$, $H_{(P)hi}l^i = 0$, we have

$$\begin{aligned}
H_{(P).i}\|_j - H_{(P).j}\|_i = H_{(P).i}|_j - H_{(P).j}|_i = \{H_{(P)hi}|_j - H_{(P)hj}|_i\}b^h \\
- g^{hk}\{H_{(P)hi}C_{.kj} - H_{(P)hj}C_{.ki}\} \qquad (5.26)
\end{aligned}$$

$$\left(\frac{L\sqrt{L\omega}}{\sqrt{t}}\right)\Big\|_j = \dot{\partial}_j\left(\frac{L\sqrt{L\omega}}{\sqrt{t}}\right)$$
$$= \frac{L^4\omega\sqrt{L\omega}}{t^{3/2}}C_{..j} + \frac{L\sqrt{L\omega}}{2\omega t^{3/2}}\{p\omega_2 + 3L\omega^2(\beta - \rho L^2)\}\, m_j \tag{5.27}$$

and

$$\overline{g}^{hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{\frac{fp}{L}} = \sqrt{\frac{L}{fp}}g^{hk}\times$$
$$\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\} - \frac{L^3\omega\sqrt{L}}{t\sqrt{fp}}\{H_{(P).i}d_{.j} - H_{(P).j}d_{.i}\}. \tag{5.28}$$

After using (3.16) and (5.12) the equation (5.28) may be written as

$$\overline{g}^{hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{\frac{fp}{L}} = \frac{L\sqrt{L\omega}}{\sqrt{t}}g^{hk}\times$$
$$\{H_{(P)hi}C_{.kj} - H_{(P)hj}C_{.ki}\} - \frac{L^4\omega\sqrt{L\omega}}{t^{3/2}}\{H_{(P).i}C_{..j} - H_{(P).j}C_{..i}\}$$
$$-\frac{L\sqrt{L\omega}}{2\omega t^{3/2}}[p\omega_2 + 3L\omega^2(\beta - \rho L^2)]\{H_{(P).i}m_j - H_{(P).j}m_i\}. \tag{5.29}$$

From (4.2), (5.26)–(5.29) it follows that equation (5.25) holds identically.

Verification of (5.23)(c) In view of (5.1) and (5.2), equation (5.23)c is equivalent to

$$\frac{L\sqrt{L\omega}(2q\omega - sp)}{2f\omega t\sqrt{p}}\{H_{(P).i}m_j - H_{(P).j}m_i\}$$
$$+\overline{g}^{hk}\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\}\sqrt{\frac{fp}{L}} = 0, \tag{5.30}$$

Since $E_{kj}l^k = 0 = E_{jk}l^k$, from (3.5), we find that the value of $\overline{g}^{hk}\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\}$ is

$$\sqrt{\frac{L}{fp}}.g^{hk}\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\} - \frac{L^3\omega\sqrt{L}}{t\sqrt{fp}}\{H_{(P).i}E_{.j} - H_{(P).j}E_{.i}\},$$

which, in view of (3.17) and (5.18), is equal to

$$-\frac{L\sqrt{L\omega}(2q\omega - sp)}{2f\omega t\sqrt{p}}\{H_{(P).i}m_j - H_{(P).j}m_i\}.$$

Hence equation (5.30) is satisfied identically.

Verification of (5.23)(d) In view of (5.1) and (5.2), equation (5.23)d is equivalent to

$$(Nm_i)\|_j - (Nm_j)\|_i + \overline{g}^{hk}(d_{hi}E_{kj} - d_{hj}E_{ki}) = 0, \tag{5.31}$$

where $N = \frac{sp-2q\omega}{2f\omega\sqrt{pt}}$.

Since $d_{hi}l^h = 0$, $E_{kj}l^k = 0$, from (3.5), we find that the value of $\overline{g}^{hk}\{d_{hi}E_{kj} - d_{hj}E_{ki}\}$ is

$$\frac{L}{fp}g^{hk}\{d_{hi}E_{kj} - d_{hj}E_{ki}\} - \frac{L^4\omega}{fpt}\{d_{.i}E_{.j} - d_{.j}E_{.i}\},$$

which, in view of (3.16), (3.17), (5.12) and (5.18), is equal to

$$-\frac{L^3(2q\omega - sp)}{2f\sqrt{p}.t^{3/2}}\{C_{..i}m_j - C_{..j}m_i\}.$$

Also,

$$(Nm_i)\|_j - (Nm_j)\|_i = N(m_i\|_j - m_j\|_i) + (\dot{\partial}_j N)m_i - (\dot{\partial}_i N)m_j.$$

Since $m_i\|_j - m_j\|_i = m_i|_j - m_j|_i = L^{-1}(l_j m_i - l_i m_j)$ and

$$\dot{\partial}_j N = -\frac{2q\omega - sp}{2Lf\omega\sqrt{pt}}l_j + \frac{L^3(sp - 2q\omega)}{2f\sqrt{p}.t^{3/2}}C_{..j},$$

we have

$$(Nm_i)\|_j - (Nm_j)\|_i = \frac{L^3(sp - 2q\omega)}{2f\sqrt{p}.t^{3/2}}(C_{..j}m_i - C_{..i}m_j). \tag{5.32}$$

Hence equation (5.31) is satisfied identically. Therefore Ricci Kühne equations of $(M_x^n, \overline{g}_x)$ given in (5.23) are satisfied.

Hence the Theorem A given in introduction is satisfied for the $\beta-$change (1.3) of Finsler metric given by $h-$vector. $\square$

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