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Equitable Coloring on Triple Star Graph Families

K.Praveena

(Department of Computer Science, Dr.G.R. Damodaran College of Science, Coimbatore-641014, Tamilnadu, India)

M.Venkatachalam

(Department of Mathematics, Kongunadu Arts and Science College, Coimbatore-641029, Tamilnadu, India)

E-mail: praveena4ks@gmail.com, venkatmaths@gmail.com

Abstract: An equitable k-coloring of a graph G is a proper k-coloring of G such that the sizes of any two color class differ by at most one. In this paper we investigate the equitable chromatic number for the Central graph, Middle graph, Total graph and Line graph of Triple star graph $K_{1,n,n,n}$ denoted by $C(K_{1,n,n,n})$, $M(K_{1,n,n,n})$, $T(K_{1,n,n,n})$ and $L(K_{1,n,n,n})$ respectively.

Key Words: Equitable coloring, Smarandachely equitable k-coloring, triple star graph, central graph, middle graph, total graph and line graph.

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§1. Introduction

A graph consist of a vertex set $V(G)$ and an edge set $E(G)$. All Graphs in this paper are finite, loopless and without multiple edges. We refer the reader [8] for terminology in graph theory. Graph coloring is an important research problem [7, 10]. A proper k-coloring of a graph is a labelling $f : V(G) \to \{1, 2, \cdots, k\}$ such that the adjacent vertices have different labels. The labels are colors and the vertices with same color form a color class. The chromatic number of a graph G, written as $\chi(G)$ is the least k such that G has a proper k-coloring.

Equitable colorings naturally arise in some scheduling, partitioning and load balancing problems [11,12]. In 1973, Meyer [4] introduced first the notion of equitable colorability. In 1998, Lih [5] surveyed the progress on the equitable coloring of graphs.

We say that a graph $G = (V, E)$ is equitably k-colorable if and only if its vertex set can be partitioned into independent sets $\{V_1, V_2, \cdots, V_k\} \subset V$ such that $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j). The smallest integer k for which G is equitable k-colorable is known as the equitable chromatic number [1,3] of G and denoted by $\chi_=(G)$. On the other hand, if V can be partitioned into independent sets $\{V_1, V_2, \cdots, V_k\} \subset V$ with $||V_i| - |V_j|| \geq 1$ holds for every pair (i, j), such a k-coloring is called a *Smarandachely equitable k-coloring*.

In this paper, we find the equitable chromatic number $\chi_=(G)$ for central, line, middle and total graphs of triple star graph.

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§2. Preliminaries

For a given graph $G = (V, E)$ we do a operation on G, by subdividing each edge exactly once and joining all the non adjacent vertices of G. The graph obtained by this process is called *central graph* of G [1] and is denoted by $C(G)$.

The *line graph* [6] of a graph G, denoted by $L(G)$ is a graph whose vertices are the edges of G and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *middle graph* [2] of G denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$ in which two vertices x, y are adjacent in $M(G)$ if the following condition hold:

(1) $x, y \in E(G)$ and x, y are adjacent in G;

(2) $x \in V(G),\ y \in E(G)$ and they are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *total graph* [1,2] of G is denoted by $T(G)$ and is defined as follows. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ is adjacent in $T(G)$, if one of the following holds:

(1) x, y are in $V(G)$ and x is adjacent to y in G;

(2) x, y are in $E(G)$ and x, y are adjacent in G;

(3) x is in $V(G)$, y is in $E(G)$ and x, y are adjacent in G.

Triple star $K_{1,n,n,n}$ [9] is a tree obtained from the double star [2] $K_{1,n,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $3n + 1$ vertices and $3n$ edges.

§3. Equitable Coloring on Central Graph of Triple Star Graph

Algorithm 1.

Input: The number 'n' of $k_{1,n,n,n}$;
Output: Assigning equitable colouring for the vetices in $C(K_{1,n,n,n})$.

begin
for $i\ =\ 1$ to n
{
$V_1 = \{e_i\}$
$C(e_i) = i;$
$V_2 = \{a_i\};$
$C(a_i) = i;$
}
$V_3 = \{v\};$
$C(v) = n + 1;$

for $i = 2$ to n

{

$V_4 = \{v_i\}$;

$C(v_i) = i - 1$;

$V_5 = \{w_i\}$;

$C(w_i) = i - 1$;

$V_6 = \{u_i\}$;

$C(u_i) = i - 1$;

}

$C(v_1) = n$;

$C(w_1) = n$;

$C(u_1) = n$;

for $i = 1$ to 5

{

$V_7 = \{s_i\}$;

$C(s_i) = n + 1$;

}

for $i = 6$ to n

{

$V_8 = \{s_i\}$;

$C(s_i) = i$;

}

$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7 \cup V_8$;

end

Theorem 3.1 *For any triple star graph $K_{1,n,n,n}$ the equitable chromatic number*

$$\chi_=[C(K_{1,n,n,n})] = n + 1.$$

Proof Let $\{v_i : 1 \leq i \leq n\}$, $\{w_i : 1 \leq i \leq n\}$ and $\{u_i : 1 \leq i \leq n\}$ be the vertices in $K_{1,n,n,n}$. The vertex v is adjacent to the vertices $v_i(1 \leq i \leq n)$. The vertices $v_i(1 \leq i \leq n)$ is adjacent to the vertices $w_i(1 \leq i \leq n)$ and the vertices $w_i(1 \leq i \leq n)$ is adjacent to the vertices $u_i(1 \leq i \leq n)$.

By the definition of central graph on $K_{1,n,n,n}$, let the edges vv_i, v_iw_i and w_iu_i $(1 \leq i \leq n)$ of $K_{1,n,n,n}$ be subdivided by the vertices $e_i, a_i, s_i(1 \leq i \leq n)$ respectively.

Clearly,

$$\begin{aligned} V[C(K_{1,n,n,n})] &= \{v\}\bigcup\{v_i : 1 \leq i \leq n\}\bigcup\{w_i : 1 \leq i \leq n\} \\ &\bigcup\{u_i : 1 \leq i \leq n\}\bigcup\{e_i : 1 \leq i \leq n\} \\ &\bigcup\{a_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\} \end{aligned}$$

The vertices v and $u_i (1 \leq i \leq n)$ induces a clique of order $n+1$ (say k_{n+1}) in $[C[K_{1,n,n,n}]]$. Therefore

$$\chi_{=}[C(K_{1,n,n,n})] \geq n+1$$

Now consider the vertex set $V[C(K_{1,n,n,n})]$ and the color class $C = \{c_1, c_2, c_3, \cdots c_{n+1}\}$. Assign an equitable coloring to $C(K_{1,n,n,n})$ by Algorithm 1. Therefore

$$\chi_{=}[C(K_{1,n,n,n})] \leq n+1.$$

An easy check shows that $||v_i| - |v_j|| \leq 1$. Hence

$$\chi_{=}[C(K_{1,n,n,n})] = n+1. \qquad \square$$

§4. Equitable Coloring on Line graph of Triple Star Graph

Algorithm 2.

Input: The number 'n' of $K_{1,n,n,n}$;
Output: Assigning equitable coloring for the vertices in $L(K_{1,n,n,n})$.

```
begin
for i = 1 to n
{
V_1 = {e_i};
C(e_i) = i;
V_2 = {s_i};
C(s_i) = i;
}
for i = 2 to n
{
V_3 = {a_i};
C(a_i) = i − 1;
}
```

$C(a_1) = n$;

$V = V_1 \cup V_2 \cup V_3$;

end

Theorem 4.1 *For any triple star graph $K_{1,n,n,n}$ the equitable chromatic number,*

$$\chi_=[L(K_{1,n,n,n})] = n.$$

Proof Let $\{v_i : 1 \leq i \leq n\}$, $\{w_i : 1 \leq i \leq n\}$ and $\{u_i : 1 \leq i \leq n\}$ be the vertices in $K_{1,n,n,n}$. The vertex v is adjacent to the vertices $v_i (1 \leq i \leq n)$ with edges $e_i (1 \leq i \leq n)$. The vertices $v_i (1 \leq i \leq n)$ is adjacent to the vertices $w_i (1 \leq i \leq n)$ with edges $a_i (1 \leq i \leq n)$. The vertices $w_i (1 \leq i \leq n)$ is adjacent to the vertices $u_i (1 \leq i \leq n)$ with edges $s_i (1 \leq i \leq n)$.

By the definition of line graph on $K_{1,n,n,n}$ the edges $e_i, a_i, s_i (1 \leq i \leq n)$ of $K_{1,n,n,n}$ are the vertices of $L(K_{1,n,n,n})$. Clearly

$$V[L(K_{1,n,n,n})] = \{e_i : 1 \leq i \leq n\} \bigcup \{a_i : 1 \leq i \leq n\} \bigcup \{s_i : 1 \leq i \leq n\}$$

The vertices $e_i (1 \leq i \leq n)$ induces a clique of order n (say K_n) in $L(K_{1,n,n,n})$. Therefore

$$\chi_=[L(K_{1,n,n,n})] \geq n.$$

Now consider the vertex set $V[L(K_{1,n,n,n})]$ and the color class $C = \{c_1, c_2, \cdots c_n\}$.

Assign an equitable coloring to $L(K_{1,n,n,n})$ by Algorithm 2. Therefore

$$\chi_=[L(K_{1,n,n,n})] \leq n.$$

An easy check shows that $||v_i| - |v_j|| \leq 1$. Hence

$$\chi_=[L(K_{1,n,n,n})] = n. \qquad \square$$

§5. Equitable Coloring on Middle and Total Graphs of Triple Star Graph

Algorithm 3.

Input: The number 'n' of $K_{1,n,n,n}$;

Output: Assigning equitable coloring for the vertices in $M(K_{1,n,n,n})$ and $T(K_{1,n,n,n})$.

begin
for $i = 1$ to n
{
$V_1 = \{e_i\}$;
$C(e_i) = i$;
$V_2 = \{s_i\}$;
$C(s_i) = i$;
}
$V_3 = \{v\}$;
$C(v) = n + 1$;
for $i = 2$ to n
{
$V_4 = \{v_i\}$;
$C(v_i) = i - 1$;
}
$C(v_1) = n$;
for $i = 3$ to n
{
$V_5 = \{a_i\}$;
$C(a_i) = i - 2$;
}
$C(a_1) = n + 1$;
$C(a_2) = n + 1$;
for $i = 4$ to n
{
$V_6 = \{w_i\}$
$C(w_i) = i - 3$;
}
$C(w_1) = n - 1$;
$C(w_2) = n$;

$C(w_3) = n+1;$

for $i = 1$ to n

{

$V_7 = \{u_i\};$

$C(u_i) = i+1;$

}

$V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7$

end

Theorem 5.1 *For any triple star graph $K_{1,n,n,n}$ the equitable chromatic number,*

$$\chi_=[M(K_{1,n,n,n})] = n+1,\ n \geq 4.$$

Proof Let $V(K_{1,n,n,n}) = \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n\}$.

By the definition of middle graph on $K_{1,n,n,n}$ each edge vv_i, v_iw_i and $w_iu_i(1 \leq i \leq n)$ in $K_{1,n,n,n}$ are subdivided by the vertices e_i, w_i, $s_i(1 \leq i \leq n)$ respectively. Clearly

$$\begin{aligned} V[M(K_{1,n,n,n})] &= \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \\ &\bigcup \{u_i : 1 \leq i \leq n\} \bigcup \{e_i : 1 \leq i \leq n\} \\ &\bigcup \{a_i : 1 \leq i \leq n\} \bigcup \{s_i : 1 \leq i \leq n\} \end{aligned}$$

The vertices v and $e_i(1 \leq i \leq n)$ induces a clique of order $n+1$ (say k_{n+1}) in $[M(K_{1,n,n,n})]$. Therefore

$$\chi_=[M(K_{1,n,n,n})] \geq n+1.$$

Now consider the vertex set $V[M(K_{1,n,n,n})]$ and the color class $C = \{c_1, c_2, \cdots c_{n+1}\}$. Assign an equitable coloring to $M(K_{1,n,n,n})$ by Algorithm 3. Therefore

$$\chi_= M[(K_{1,n,n,n})] \leq n+1, \quad ||v_i| - |v_j|| \leq 1.$$

Hence

$$\chi_=[M(K_{1,n,n,n})] = n+1 \ \forall n \geq 4.$$ □

Theorem 5.2 *For any triple star graph $K_{1,n,n,n}$ the equitable chromatic number,*

$$\chi_=[T(K_{1,n,n,n})] = n+1, \quad n \geq 4.$$

Proof Let $V(K_{1,n,n,n}) = \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n\}$ and $E(K_{1,n,n,n}) = \{e_i : 1 \leq i \leq n\} \bigcup \{a_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

By the definition of Total graph, the edge vv_i, v_iw_i and $w_iu_i(1 \leq i \leq n)$ of $K_{1,n,n,n}$ be subdivided by the vertices e_i, a_i and $s_i(1 \leq i \leq n)$ respectively. Clearly

$$\begin{aligned} V[T(K_{1,n,n,n})] &= \{v\} \bigcup \{v_i : 1 \leq i \leq n\} \bigcup \{w_i : 1 \leq i \leq n\} \\ & \bigcup \{u_i : 1 \leq i \leq n\} \bigcup \{e_i : 1 \leq i \leq n\} \\ & \bigcup \{a_i : 1 \leq i \leq n\} \bigcup \{s_i : 1 \leq i \leq n\}. \end{aligned}$$

The vertices v and $e_i(1 \leq i \leq n)$ induces a clique of order $n+1$ (say k_{n+1}) in $T(K_{1,n,n,n})$. Therefore

$$\chi_{=}[T(K_{1,n,n,n})] \geq n+1, \quad n \geq 4.$$

Now consider the vertex set $V(T(K_{1,n,n,n}))$ and the color class $C = \{c_1, c_2, \cdots, c_{n+1}\}$. Assign an equitable coloring to $T(K_{1,n,n,n})$ by Algorithm 3. Therefore

$$\chi_{=}[T(K_{1,n,n,n})] \leq n+1, \quad n \geq 4, \quad ||v_i| - |v_j|| \leq 1.$$

Hence

$$\chi_{=}[T(K_{1,n,n,n})] = n+1, \forall n \geq 4. \qquad \square$$

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