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Even Modular Edge Irregularity Strength of Graphs

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Abstract: A new graph characteristic, even modular edge irregularity strength of graphs is introduced. Estimation on this parameter is obtained and the precise values of this parameter are obtained for some families of graphs.

Key Words: Irregular labeling, modular irregular labeling, even modular edge irregular labeling, vertex k-labeling, irregularity strength, modular irregularity strength, even modular edge irregularity strength, Smarandachely p-modular edge irregularity strength.

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§1. Introduction

Let $G = (V, E)$ be a simple graph, having at most one isolated vertex and no component of order 2. A map that carries vertex set (edge set or both) as domain to the positive integers $\{1, 2, \cdots, k\}$ is called *vertex k-labeling* (*edge k-labeling or total k-labeling*). Well-known parameter irregularity strength of a graph introduced by Chartrand et al. [6]. A simple graph G is called irregular if there exists an edge k-labeling $\lambda : E(G) \to \{1, 2, \cdots, k\}$ such that the weight of a vertex v under the labeling defined by $w_\lambda(v) = \sum \lambda(uv)$, are pairwise distinct. The minimum value of k, for which G is irregular, called irregularity strength of G denoted by $s(G)$.

The parameter irregularity strength of a graph is attracted by numerous authors. Aigner and Triesh [1] proved that $\mathrm{s}(G) \leqslant n - 1$ if G is a connected graph of order n, and $\mathrm{s}(G) \leqslant n + 1$ otherwise. Nierhoff [15] refined their method and showed that $\mathrm{s}(G) \leqslant n - 1$ for all graphs with finite irregularity strength, except for K_3. This bound is tight e.g. for stars. In particular Faudree and Lehel [8] showed that if G is d-regular ($d \geqslant 2$), then $\left\lceil \frac{n+d-1}{d} \right\rceil \leqslant \mathrm{s}(G) \leqslant \left\lceil \frac{n}{2} \right\rceil + 9$, and they conjectured that $\mathrm{s}(G) \leqslant \left\lceil \frac{n}{d} \right\rceil + c$ for some constant c. Przybylo in [16] proved that $\mathrm{s(G)} \leqslant 16\frac{\mathrm{n}}{\mathrm{d}} + 6$. Kalkowski, Karonski and Pfender [12] showed that $\mathrm{s}(G) \leqslant 6\frac{n}{\delta} + 6$, where δ is the minimum degree of graph G. Currently Majerski and Przybylo [13] proved that $\mathrm{s}(G) \leqslant (4 + o(1))\frac{n}{\delta} + 4$ for graphs with minimum degree $\delta \geqslant \sqrt{n} \ln n$. Other interesting results on the irregularity strength can be found in [3, 4, 5, 7, 9]. For recent survey of graph labeling refer the paper [10].

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Ali Ahmad et al.[2] introduced edge irregularity strength of a graph as follows: Consider a simple graph G together with a vertex k-labeling $\chi : V(G) \to \{1, 2, \cdots, k\}$. The weight of an edge xy in G, denoted by $wt(xy) = \chi(x) + \chi(y)$. A vertex k-labeling is defined to be an edge irregular k-labeling of the graph G if for every two different edges e and f there is $wt(e) \neq wt(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by $es(G)$. The lower bound of $es(G)$ was given by the following inequality

$$es(G) \geqslant max\{\left\lceil \frac{E(G)+1}{2} \right\rceil, \Delta\}$$

where Δ is the maximum degree of graph G. Ibrahim Tarawneh et al. [11], determined the exact value of edge irregularity strength of corona graphs of path P_n with P_2, P_n with K_1 and P_n with S_m.

Martin Bača et al. [14] introduced modular irregularity strength of a graph. An edge labeling $\psi : E(G) \to \{1, 2, \cdots, k\}$ is called *modular irregular k-labeling* if there exists a bijective weight function $\sigma : V(G) \to Z_n$ defined by $\sigma(x) = \sum \psi(xy)$ called *modular weight* of the vertex x, where Z_n is the group of integers modulo n and the sum is over all vertices y adjacent to x. They defined the *modular irregularity strength* of a graph G, denoted by ms(G), as the minimum k for which G has a modular irregular k-labeling.

Motivated by the edge irregularity strength of graphs we introduce a new parameter, an even modular edge irregularity strength of graph, a modular version of edge irregularity strength.

Let $G = (V, E)$ be a (n, m)-graph together with a vertex k-labeling $\rho : V(G) \to \{1, 2, \cdots, k\}$. Define a set of edge weight $W = \{wt(uv) : wt(uv) = \rho(u) + \rho(v), \forall\ uv \in E\}$. Vertex labeling ρ is called *even modular edge irregular labeling* if there exists a bijective map $\sigma : W \to M$ defined for each edge weight $wt(uv)$ there corresponds an element $x \in M$ such that $wt(uv) \equiv x$ (mod $2m$), where $M = \{0, 2, 4, \cdots, 2(m-1)\}$. We define the *even modular edge irregularity strength* of a graph G, denoted by emes(G), as the minimum k for which G has an even modular edge irregular labeling. If there doesn't exist an even modular edge irregular labeling for G, we define emes$(G) = \infty$. Generally, if $M = \{0, p, 2p, \cdots, (m-1)p\}$ for a prime number p, such a modular edge irregular labeling is called a *Smarandache p-modular edge irregular labeling* and the minimum k for which G has a Smarandachely p-modular edge irregular labeling is denoted by emes$^p(G)$. Clearly, emes$^2(G) =$ emes(G).

The main aim of this paper is to show a lower bound of the even modular edge irregularity strength and determine the precise values of this parameter for some families of graphs.

§2. Main Results

Following theorem gives the lower bound of even modular edge irregularity strength of a graph.

Theorem 2.1 *Let G be a (n,m)-graph. Then $emes(G) \geq m$.*

Proof Let G be a (n,m)-graph together with an even modular edge irregular labeling

$\rho : V(G) \to \{1, 2, \cdots, k\}$. Consider the even edge weights of G, there should be an edge e such that $wt(e) \equiv 0 \pmod{2m}$. Since the weight of e must be at least $2m$, $emes(G) \geq m$. □

Lemma 2.1 *Let $(d_1, d_2, \cdots, d_n)$ be the degree sequence of a graph G and let $(l_1, l_2, \ldots, l_n)$ be the corresponding vertex labels of an even modular edge irregular labeling of G. Then the sum of all the edge weights denoted as S is equal to the sum of the product of degree with its corresponding labels, that is,*

$$S = \sum_{e \in E} wt(e) = \sum_{i=1}^{n} d_i l_i.$$

Lemma 2.2 *In any even modular edge irregular labeling of C_n, labels of all vertices are of same parity.*

Proof By definition, weight of an edge is sum of the labels of its end vertices. To obtain an even edge weight, both the labels must be either odd or even, and hence all the vertex labels of C_n are of same parity. □

Theorem 2.2 *Let C_n be a cycle of order $n \geq 3$. Then*

$$emes(C_n) = \begin{cases} n+1, & \text{if } n \equiv 0 \pmod 4, \\ n, & \text{if } n \equiv 1 \pmod 4, \\ n+2, & \text{if } n \equiv 3 \pmod 4, \\ \infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof Let $V(C_n) = \{v_i : i = 1, 2, \cdots, n\}$ be the vertex set and let $E(C_n) = \{e_i = v_i v_{i+1} : i = 1, 2, \cdots, n\}$ be the edge set of the cycle C_n. Define the vertex labeling $\rho : V \to \{1, 2, \cdots, n+2\}$ as follows:

$$\rho(v_i) = 2i - 1, \ 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$$

If $n \equiv 0, 1 \pmod 4$, then, for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$\rho(v_{n+1-i}) = \begin{cases} 2i - 1, & \text{i is odd} \\ 2i + 1, & \text{i is even} \end{cases}$$

If $n \equiv 3 \pmod 4$, then for $2 \leq i \leq \lceil \frac{n}{2} \rceil$,

$$\rho(v_{n+2-i}) = \begin{cases} 2i - 1, & \text{i is odd} \\ 2i + 1, & \text{i is even} \end{cases}$$

We can easily check that the above labeling ρ, is an even modular edge irregular labeling of C_n. Thus,

$$emes(C_n) \le \begin{cases} n+1, & \text{if } n \equiv 0 \pmod 4 \\ n, & \text{if } n \equiv 1 \pmod 4 \\ n+2, & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Now let us find the lower bound of $emes(C_n)$ as follows:

Case 1. Suppose $n \equiv 0 \pmod 4$. Consider the set of even edge weights $W(C_n) = \{2, 4, 6, \cdots, 2n\}$. To obtain the weight 2 for an edge, we must assign label 1 to both of its end vertices, and hence all the vertices of C_n must receive odd labels by Lemma 2.2. Since the heaviest weight is $2n$, $emes(C_n) \ge n+1$.

Case 2. Suppose $n \equiv 1 \pmod 4$. By Theorem 2.1, $emes(C_n) \ge n$.

Case 3. Suppose $n \equiv 3 \pmod 4$. Assume that the cycle C_n has the set of even edge weights $W(C_n) = \{2, 4, \cdots, 2n\}$, then $\frac{S}{2}$ is even, where S is the sum of the weights. Since the least weight is 2, by Lemma 2.2 all the vertex labels of C_n must be odd and hence $\sum_{i=1}^{n} l_i$ is odd, which is a contradiction to $\sum_{i=1}^{n} l_i = \frac{S}{2}$ by Lemma 2.1.

Assume that C_n has the set of even edge weights $W(C_n) = \{4, 6, \cdots, 2n+2\}$. Now the sum of the labels,

$$\sum_{i=1}^{n} l_i = \frac{S}{2} = \frac{(n+1)(n+2)}{2} - 1$$

is odd and hence each label must odd. Heaviest weight $2n+2$ can be obtained by assigning the label at least $n+2$. Thus, $emes(C_n) \ge n+2$.

Case 4. Suppose $n \equiv 2 \pmod 4$. If the cycle C_n has an even modular edge irregular labeling, then the sum of the edge weights $S \equiv 2 \pmod 4$, and hence $\frac{S}{2}$ is odd. When $n \equiv 2 \pmod 4$, sum of the labels $\sum_{i=1}^{n} l_i$ is even, which is a contradiction to $\sum_{i=1}^{n} l_i = \frac{S}{2}$. Thus, $emes(C_n) = \infty$, if $n \equiv 2 \pmod 4$. □

Theorem 2.3 *Let P_n be a path of order $n \ge 2$. Then*

$$emes(P_n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even.} \end{cases}$$

Proof Let $V(P_n) = \{v_i : i = 1, 2, \cdots, n\}$ be the vertex set and let $E(P_n) = \{e_i = v_i v_{i+1} : i = 1, 2, \cdots, n\}$ be the edge set of the path P_n.

Define the vertex n-labeling $\theta : V \to \{1, 2, \cdots, n\}$ as follows:

For $1 \le i \le n$,

$$\theta(v_i) = \begin{cases} i, & \text{i is odd} \\ i-1, & \text{i is even.} \end{cases}$$

Clearly, θ is an even modular edge irregular labeling of P_n. Thus, the upper bound of

$emes(P_n)$ can be obtained as follows:

$$emes(P_n) \leq \begin{cases} n, & \text{if n is odd} \\ n-1, & \text{if n is even.} \end{cases}$$

Let us find the lower bound of $emes(P_n)$.

Case 1. Assume that n is odd. Consider the optimal even edge weight $W(P_n) = \{2, 4, \cdots, 2(n-1)\}$. Since the least weight is 2, all the vertices of P_n must receive odd labels. To obtain the heaviest weight $2(n-1)$, we must assign vertex label at least n. Thus, $emes(P_n) \geq n$.

Case 2. Assume that n is even. In this case, the lower bound can be obtain directly from Theorem 2.1. □

Theorem 2.4 *Let $K_{1,n}$ be a star graph of order $n+1, n \geq 1$. Then $emes(K_{1,n}) = 2n-1$.*

Proof Let $V(K_{1,n}) = \{x, v_i : i = 1, 2, \cdots, n\}$ be the vertex set and let $E(K_{1,n}) = \{e_i = xv_i : i = 1, 2, \cdots, n\}$ be the edge set of the path $K_{1,n}$.

Define the vertex labeling $\lambda_1 : V \to \{1, 2, \cdots, 2n-1\}$ as follows:

$\lambda_1(x) = 1,$

$\lambda_1(v_i) = 2i - 1, 1 \leq i \leq n.$

From the above even modular edge irregular labeling λ_1, upper bound of $emes(K_{1,n})$ is obtained as follows, $emes(K_{1,n}) \leq 2n-1$.

Consider the optimal even edge weights $W(K_{1,n}) = \{2, 4, \ldots, 2n\}$. Since the least weight is 2, the vertex x must be label with 1. To obtain the heaviest weight $2n$, we must assign label at least $2n-1$ to other end vertex. Thus, $emes(K_{1,n}) \geq 2n-1$. Hence the theorem. □

Theorem 2.5 *Let $K_{2,n}$ be the complete bipartite graph of order $n+2, n \geq 2$. Then $emes(K_{2,n}) = 2n+1$.*

Proof Let $V(K_{2,n}) = \{x, y, v_i : i = 1, 2, \cdots, n\}$ be the vertex set and let $E(K_{2,n}) = \{xv_i, yv_i : i = 1, 2, \cdots, n\}$ be the edge set of the complete bipartite graph $K_{2,n}$.

Define the vertex labeling $\lambda_2 : V \to \{1, 2, \cdots, 2n+1\}$ as follows:

$\lambda_2(x) = 1, \quad \lambda_2(y) = 2n+1$

$\lambda_2(v_i) = 2i - 1, 1 \leq i \leq n.$

From the above even modular edge irregular labeling λ_2, upper bound of $emes(K_{2,n})$ is obtained as follows: $emes(K_{2,n}) \leq 2n+1$.

Consider the even edge weights of $K_{2,n}$ as $2, 4, \cdots, 4n$. Since the least edge weight is 2, all the vertices must receive odd labels. Therefore, we must assign label at least $2n+1$, to obtain the heaviest weight $4n$. Hence $emes(K_{2,n}) \geq 2n+1$. □

A rectangular graph R_n, $n \geq 2$, is a graph obtained from the path P_{n+1} by replacing each edge of the path by a rectangle C_4. Let

$$V(R_n) = \{v_i : i = 1, 2, \cdots, 2n\} \bigcup \{u_j : j = 1, 2, \cdots, n+1\}$$

be the vertex set and let

$$\begin{aligned} E(R_n) &= \{v_{2i-1}v_{2i} : i = 1, 2, \cdots, n\} \bigcup \{u_i u_{i+1} : i = 1, 2, \cdots, n\} \\ & \bigcup \{v_{2i-1}u_i : i = 1, 2, \cdots, n\} \bigcup \{v_{2i-2}u_i : i = 2, 3, \cdots, n+1\} \end{aligned}$$

be the edge set of the the rectangular graph R_n. The following theorem gives the precise value of even modular edge irregularity strength of rectangular graph.

Theorem 2.6 *Let R_n be a rectangular graph of order $3n+1$, $n \geq 2$. Then $emes(R_n) = 4n+1$.*

Proof Define the vertex labeling $\alpha : V \to \{1, 2, \cdots, 4n+1\}$ as follows:

$\alpha(v_i) = 2i - 1, 1 \leq i \leq 2n,$
$\alpha(u_i) = 4i - 3, 1 \leq i \leq n+1.$

Upper bound $emes(R_n) \leq 4n+1$ can be obtained from the above labeling α.

Consider the even edge weights of R_n as $2, 4, \cdots, 8n$. Since the least weight is 2, all the vertex labels must be odd. Therefore, we must assign label at least $4n+1$, to obtain the heaviest weight $8n$. Hence, $emes(R_n) \geq 4n+1$. □

Theorem 2.7 *Let tP_4, $t \geq 1$, denote the disjoint union of t copies of path P_4. Then $emes(tP_4) = 3t$.*

Proof Let $V(tP_4) = \{u_{ij} : 1 \leq i \leq t, 1 \leq j \leq 4\}$ be the vertex set and let $E(tP_4) = \{u_{i1}u_{i2},\ u_{i2}u_{i3},\ u_{i3}u_{i4} : 1 \leq i \leq t\}$ be the edge set of tP_4. Define the vertex labeling $\beta : V \to \{1, 2, \cdots, 3t\}$ as follows:

$\beta(u_{i1}) = \beta(u_{i2}) = 3i - 2,\ 1 \leq i \leq t,$
$\beta(u_{i3}) = \beta(u_{i4}) = 3i,\ 1 \leq i \leq t.$

Clearly, β is an even modular edge irregular labeling of tP_4 and hence $emes(tP_4) \leq 3t$. The lower bound $emes(tP_4) \geq 3t$ can be obtained directly from Theorem 2.1. Hence, we get that $emes(tP_4) = 3t$. □

Theorem 2.8 *Let tC_3, $t \geq 2$, denote the disjoint union of t copies of cycle C_3. Then $emes(tC_3) = 3t+2$.*

Proof Let $V(tC_3) = \{v_{ij} : 1 \leq i \leq t, 1 \leq j \leq 3\}$ be the vertex set and let $E(tC_3) = \{v_{i1}v_{i2}, v_{i2}v_{i3}, v_{i1}v_{i3} : 1 \leq i \leq t\}$ be the edge set of tC_3. Define the vertex labeling $\theta : V \to \{1, 2, \cdots, 3t+2\}$ as follows:

$$\theta(v_{i1}) = \begin{cases} 1, & i = 1 \\ 3t, & 2 \leq i \leq t, \end{cases} \qquad \theta(v_{i2}) = 3i + 2, 1 \leq i \leq t, \qquad \text{and} \qquad \theta(v_{i3}) = \begin{cases} 3, & i = 1 \\ 3i+1, & 2 \leq i \leq t, \end{cases}$$

Clearly, θ is an even modular edge irregular labeling of tP_4 and hence $emes(tC_3) \leq 3t+2$.

Consider the optimal edge weights of tC_3 as $4, 6, 8, \cdots, 6t+2$. Since any two adjacent vertices of tC_3 can not receive the same labels, we must assign label at least $3t+2$ to get the heaviest label $6t+2$. Hence, $emes(tC_3) \geq 3t+2$. □

Ladder graph $L_n = K_2 \times P_n, n \geq 3$ is formed by taking two isomorphic copies of P_n and joining the corresponding vertices by an edge. Let $V = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set and let

$$E = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_i v_i : 1 \leq i \leq n\}$$

be the edge set of L_n. The following theorem gives the precise value of even modular edge irregularity strength of ladder graph.

Theorem 2.9 *Let $L_n = K_2 \times P_n, n \geq 3$ be the ladder graph. Then*

$$emes(L_n) = \begin{cases} 3n-2, & \text{if } n \text{ is odd,} \\ 3n-1, & \text{if } n \text{ is even.} \end{cases}$$

Proof Defined the vertex labeling $\phi : V \to \{1, 2, \cdots, 3n-1\}$ as follows:

$$\phi(u_i) = \begin{cases} 3i-2, & \text{if i is odd} \\ 3i-3, & \text{if i is even} \end{cases} \qquad 1 \leq i \leq n,$$

$$\phi(v_i) = \begin{cases} 3i-2, & \text{if i is odd} \\ 3i-1, & \text{if i is even.} \end{cases} \qquad 1 \leq i \leq n.$$

Clearly, ϕ is an even modular edge irregular labeling of L_n and hence

$$emes(L_n) \leq \begin{cases} 3n-2, & \text{if n is odd,} \\ 3n-1, & \text{if n is even.} \end{cases}$$

Lower bound $emes(L_n) \geq 3n-2$, can be obtained directly from Theorem 2.1, when n is odd.

Suppose n is even. Consider optimal edge weights of L_n as $2, 4, \cdots, 6n-4$. Since L_n has a span cycle, all the vertices of L_n must receive the labels of same parity. Furthermore, to obtain the edge weight 2, the corresponding end vertices must be label 1, and hence all the labels must be odd. Thus $emes(L_n) \geq 3n-1$. Hence the theorem. □

§3. Conclusion

In this paper we introduced a new graph parameter, the even modular edge irregularity strength, $emes(G)$, as a modular version of edge irregularity strength. We determined the exact value of even modular edge irregularity strength of some families of graphs and a lower bound of *emes* is obtained. However, the determination of upper bound is still open.

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