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Finslerian Hypersurfaces and Quartic Change of Finsler Metric

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Abstract: In the present paper we have studied the Finslerian hypersurfaces and quartic change of a Finsler metric. The relationship with Finslerian hypersurface and the other which is finslerian hypersurface given by quartic change have been obtained. We have also proved that quartic change makes three type of hyper surfaces invarient under certain condition. These three type of hyper surfaces are hyperplanes of first, second, and third kind.

Key Words: Finsler metric, Finslerian hyperspaces, quartic change, hyperplanes of first, second, and third kind.

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§1. Introduction

Let (M^n, L) be an n- dimensional Finsler space on a differential Manifold M^n, equipped with fundamental function $L(x, y)$. In 1984 C. Shibata [11] introduced the transformation of Finsler metric:

$$L^*(x,y) = f(L, \beta), \tag{1.1}$$

where $\beta = b_i y^i$, $b_i(x)$ are the components of a covariant vector in (M^n, L) and f is positively homogeneous function of degree one in L and β. The change of metric is called a β- change. A particular β- change of a Finsler metric function is a quartic change of metric function is defined as

$$L^* = (L^4 + \beta^4)^{1/4}. \tag{1.2}$$

If $L(x, y)$ reduces to the metric function of Riemann space then $L^*(x, y)$ reduces to the metric function of space generated by quartic metric. Due to this reason this transformation (1.2) has been called the quartic change of Finsler metric.

On the other hand, in 1985, M. Matsumoto investigated the theory of Finslerian hypersurface [4]. He has defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds.These names were given by Rapesak [8]. Kikuchi [3] gave other name following haimovichi [1]. In the year 2005, Prasad and Tripathi [7] studied the Finslerian Hy-

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persurfaces and Kropina change of a Finsler metric and obtained different results in his paper. Again, in the year 2005, Prasad, Chaubey and Patel [6] studied the Finslerian Hypersurfaces and Matsumoto change of a Finsler metric and obtained different results.

In the present paper, using the field of linear frame ([2, 3, 5]) we shall consider Finslerian hypersurfaces given by a quartic change of a Finsler metric.Our purpose is to give some relations between the original Finslerian hypersurface and the other which is Finselrian hypersurface given by quartic change. We also show that a quartic change makes three types of hypersurfaces invariant under certain condition.

§2. **Preliminaries**

Let M^n be an n-dimensional smooth manifold and $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with the fundamental function $L(x, y)$ on M^n. The metric tensor $g_{ij}(x, y)$ and Cartan's C-tensor $C_{ijk}(x, y)$ are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

respectively and we can introduce the Cartan's Connection
$C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ in F^n.

A hypersurface M^{n-1} of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where u^α are Gaussian coordinates on M^{n-1} and Greek indices vary from 1 to $n-1$. Here we shall assume that the matrix consisting of the projection factors $B^i_\alpha = \partial x^i/\partial u^\alpha$ is of rank $n-1$. The following notations are also employed: $B^i_{\alpha\beta} = \partial^2 x^i/\partial u^\alpha \partial u^\beta$, $B^i_{o\beta} = v^\alpha B^i_{\alpha\beta}$. If the supporting element y^i at a point (u^α) of M^{n-1} is assumed to be tangential to M^{n-1}, we may then write $y^i = B^i_\alpha(u)v^\alpha$ i.e. v^α is thought of as the supporting element of M^{n-1} at the point (u^α). Since the function $\bar{L}(u, v) = L(x(u), y(u, v))$ gives to a Finsler metric of M^{n-1}, we get a $n-1$ -dimensional Finsler space $F^{n-1} = (M^{n-1}, \bar{L}(u, v))$.

At each point (u^α) of F^{n-1}, the unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}B^i_\alpha N^j = 0, \quad g_{ij}N^i N^j = 1. \tag{2.1}$$

If (B^α_i, N_i) is the inverse matrix of (B^i_α, N^i), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0, \quad N^i N_i = 1 \; and \; B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \quad N_i = g_{ij}N^j. \tag{2.2}$$

For the induced Cartan's connection $IC\Gamma = (F^\alpha_{\beta\gamma}, N^\alpha_\beta, C^\alpha_{\beta\gamma})$ on F^{n-1}, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature vector H_α are respectively given by [8]

$$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta) + M_\alpha H_\beta, \quad H_\beta = N_i(B^i_{0\beta} + N^i_j B^j_\beta), \tag{2.3}$$

where

$$M_\alpha = C_{ijk}B^i_\alpha N^j N^k. \tag{2.4}$$

Contracting $H_{\alpha\beta}$ by v^α, we immediately get $H_{0\beta} = H_{\alpha\beta}v^\alpha = H_\beta$. Furthermore the second fundamental v-tensor $M_{\alpha\beta}$ is given by [10]

$$M_{\alpha\beta} = C_{ijk}B^i_\alpha B^j_\beta N^k. \tag{2.5}$$

§3. Quartic Changed Finsler Space

Let $F^n = (M^n, L)$ be a given Finsler space and let $b_i(x)dx^i$ be a one form on M^n. We shall define on M^n a function $L^*(x,y)(> 0)$ by the equation (1.2) where we put $\beta(x,y) = b_i(x)y^i$. To find the metric tensor g^*_{ij}, the angular metric tensor h^*_{ij}, the Cartan's C-tensor C^*_{ijk} of $F^{*n} = (M^n, L^*)$ we use the following results.

$$\partial\beta/\partial y^i = b_i, \quad \partial L/\partial y^i = l_i \quad \partial l_j/\partial y^i = L^{-1}h_{ij}, \tag{3.1}$$

where h_{ij} are components of angular metric tensor of F^n given by

$$h_{ij} = g_{ij} - l_i l_j = L\partial^2 L/\partial y^i \partial y^j.$$

The successive differentiation of (1.2) with respect to y^i and y^j give

$$l^*_i = \frac{L^3 l_i + \beta^3 b_i}{(L^4 + \beta^4)^{3/4}} \tag{3.2}$$

$$\begin{aligned} h^*_{ij} \quad = \quad & \frac{1}{(L^4+\beta^4)^{3/2}}[L^2(L^4+\beta^4)h_{ij} + 3\beta^4 L^2 l_i l_j \\ & +3L^4\beta^2 b_i b_j - 3L^3\beta^3(l_i b_j + l_j b_i)] \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we get the following relation between metric tensors of F^n and F^{*n}

$$\begin{aligned} g^*_{ij} \quad = \quad & \frac{1}{(L^4+\beta^4)^{3/2}}[L^2(L^4+\beta^4)g_{ij} + 2\beta^4 L^2 l_i l_j \\ & +\beta^2(3L^4+\beta^4)\beta^2 b_i b_j - 2L^3\beta^3(l_i b_j + l_j b_i)]. \end{aligned} \tag{3.4}$$

Differentiating (3.4) with respect to y^k and using (3.1) we get the following relation between the Cartan's C-tensor of F^n and F^{*n}

$$\begin{aligned} C^*_{ijk} &= \Big(\frac{L^2}{\sqrt{L^4+\beta^4}}\Big)C_{ijk} - \Big(\frac{L^2\beta^3}{(L^4+\beta^4)^{3/2}}\Big)(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) \\ &\quad -3\Big(\frac{(\beta^4-L^4)}{(L^4+\beta^4)^{5/2}}\Big)m_i m_j m_k, \end{aligned} \tag{3.5}$$

where $m_i = b_i - (\beta/L)l_i$. It is to be noted that

$$m_i l^i = 0,\ m_i b^i = b^2 - \beta^2/L^2,\ h_{ij}l^j = 0, h_{ij}m^j = h_{ij}b^i = m_i, \tag{3.6}$$

where $m^i = g^{ij}m_j = b^i - (\beta/L)l^i$.

§4. Hypersurface Given by a Quartic Change

Consider a Finslerian hypersurface $F^{n-1} = (M^{n-1}, \bar{L}(u,v))$ of the F^n and another Finslerian hypersurface $F^{*n-1} = (M^{n-1}, \bar{L}^*(u,v))$ of the F^{*n} given by the quartic change. Let N^i be the unit normal vector at each point of F^{n-1} and (B^α_i, N_i) be the inverse matrix of (B^i_α, N^i). The functions B^i_α may be considered as components of $n-1$ linearly independent tangent vectors of F^{n-1} and they are invariant under quartic change. Thus we shall show that a unit normal vector $N^{*i}(u,v)$ of F^{*n-1} is uniquely determined by

$$g^*_{ij}B^i_\alpha N^{*j} = 0, \quad g^*_{ij}N^{*i}N^{*j} = 1. \tag{4.1}$$

Contrtacting (3.4) by N^iN^j and paying attention to (2.1) and $l_iN^i = 0$, we have

$$g^*_{ij}N^iN^j = \frac{L^2(L^4+\beta^4) + \beta^2(3L^4+\beta^4)(b_iN^i)^2}{(L^4+\beta^4)^{3/2}}.$$

Therefore we obtain

$$g^*_{ij}\Big(\pm\frac{(L^4+\beta^4)^{3/4}N^i}{\sqrt{L^2(L^4+\beta^4)+\tau(b_iN^i)^2}}\Big)\Big(\pm\frac{(L^4+\beta^4)^{3/4}N^j}{\sqrt{L^2(L^4+\beta^4)+\tau(b_iN^i)^2}}\Big) = 1.$$

where

$$\tau = \beta^2(3L^4+\beta^4).$$

Hence we can put

$$N^{*i} = \frac{(L^4+\beta^4)^{3/4}N^i}{\sqrt{L^2(L^4+\beta^4)+\tau(b_iN^i)^2}}, \tag{4.2}$$

where we have choosen the positive sign in order to fix an orientation.

Using (2.1), (3.4), (4.1) and (4.2) we obtain from the first condition of (4.1),

$$[-\frac{2L^3}{(L^4+\beta^4)^{3/2}}l_iB^i_\alpha+\frac{\beta^2(3L^4+\beta^4)}{(L^4+\beta^4)^{3/2}}l_iB^i_\alpha]\frac{(L^4+\beta^4)^{3/4}b_jN^j}{\sqrt{L^2(L^4+\beta^4)+\tau(b_iN^i)^2}}=0$$

If

$$-2L^3\beta l_iB^i_\alpha+(3L^4+\beta^4)b_iB^i_\alpha=0$$

then contracting it by v^α and using $y^i=B^i_\alpha v^\alpha$ we get $L=0$ which is a contradiction with assumption that $L>0$. Hence $b_iN^i=0$. Therefore (4.2) is rewritten as

$$N^{*i}=\frac{(L^4+\beta^4)^{1/4}N^i}{L}. \tag{4.3}$$

Summary the above, we obtain

Proposition 4.1 *For a field of linear frame $(B^i_1, B^i_2, \cdots, B^i_{n-1}, N^i)$ of F^n there exists a field of linear frame $(B^i_1, B^i_2, \cdots, B^i_{n-1}, N^{*i}=\frac{(L^4+\beta^4)^{1/4}N^i}{L})$ such that* (4.1) *is satisfied along F^{*n-1} and then b_i is tangential to both the hypersurface F^{n-1} and F^{*n-1}.*

The quantities $B^{*\alpha}_i$ are uniquely defined along F^{n-1} by

$$B^{*\alpha}_i=g^{*\alpha\beta}g^*_{ij}B^j_\beta,$$

where $(g^{*\alpha\beta})$ is the inverse matrix of $(g^*_{\alpha\beta})$. Let $(B^{*\alpha}_i, N^*_i)$ be the inverse of (B^i_α, N^i), then we have $B^i_\alpha B^{*\beta}_i=\delta^\beta_\alpha, B^i_\alpha N^*_i=0, N^{*i}N^*_i=1$ and furthermore $B^i_\alpha B^{*\alpha}_j+N^{*i}N^*_j=\delta^i_j$. We also get $N^*_i=g^*_{ij}N^{*j}$ which in view of $(3.4),(2.2)$ and (4.3) gives

$$N^*_i=\frac{L}{(L^4+\beta^4)^{1/4}}N_i. \tag{4.4}$$

We denote the Cartan's connection of F^n and F^{*n} by $(F^i_{jk}, N^i_j, C^i_{jk})$ and $(F^{*i}_{jk}, N^{*i}_j, C^{*i}_{jk})$ respectively and put $D^i_{jk}=F^{*i}_{jk}-F^i_{jk}$ which will be called difference tensor. We choose the vector field b_i in F^n such that

$$D^i_{jk}=A_{jk}b^i-B_{jk}l^i, \tag{4.5}$$

where A_{jk} and B_{jk} are components of a symmetric covariant tensor of second order. Since $N_ib^i=0$ and $N_il^i=0$, from (4.5) we get

$$N_iD^i_{jk}=0,\quad N_iF^{*i}_{jk}=N_iF^i_{jk},\quad \text{and}\quad N_iD^i_{0k}=0. \tag{4.6}$$

Therefore from (3.3) and (4.3) we get

$$H^*_\alpha=\frac{L}{(L^4+\beta^4)^{1/4}}H_\alpha. \tag{4.7}$$

If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of enveloping space F^n, then F^{n-1} is called a hyperplane of the first kind [?]. A hyperplane of the first kind is characterized by $H_\alpha = 0$. Hence from (4.7), we have

Theorem 4.1 *If $b_i(x)$ be a vector field in F^n satisfying* (4.5)*, then a hypersurface F^{n-1} is a hyperplane of the first kind if and only if the hypersurface F^{*n-1} is a hyperplane of the first kind.*

Next contracting (3.5) by $B^\alpha_i N^{*j} N^{*k}$ and paying attention to (4.4), $m_i N^i = 0, h_{jk} N^j N^k = 1$ and $h_{ij} B^i_\alpha N^j = 0$ we get

$$M^*_\alpha = M_\alpha - \frac{\beta^3}{(L^4+\beta^4)^{3/2}} m_i B^i_\alpha. \tag{4.8}$$

From (2.3),(4.4) (4.5), (4.6), (4.7) and (4.8), we have

$$H^*_{\alpha\beta} = \frac{L}{(L^4+\beta^4)^{1/4}}(H_{\alpha\beta} - \frac{\beta^3}{(L^4+\beta^4)^{3/2}} m_i B^i_\alpha H_\beta). \tag{4.9}$$

If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also h-path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the second kind [?]. A hyperplane of the second kind is characterized by $H_{\alpha\beta} = 0$. Since $H_{\alpha\beta} = 0$ implies that $H_\alpha = 0$, from (4.7) and (4.9) we have the following

Theorem 4.2 *If $b_i(x)$ be a vector field in F^n satisfying* (4.5)*, then a hypersurface F^{n-1} is a hyperplane of the second kind if and only if the hypersurface F^{*n-1} is a hyperplane of the second kind.*

Finally contrcting (3.5) by $B^i_\alpha B^j_\beta N^{*k}$ and paying attention to (4.3) we have

$$M^*_{\alpha\beta} = \frac{L}{(L^4+\beta^4)^{1/4}} M_{\alpha\beta}. \tag{4.10}$$

If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1}, the F^{n-1} is called a hyperplane of the third kind [8]. A hyperplane of the third kind is characterized by $H_{\alpha\beta} = 0, M_{\alpha\beta} = 0$. From (4.7), (4.9) and (4.10) we have

Theorem 4.3 *If $b_i(x)$ be a vector field in F^n satisfying* (4.5)*, then a hypersurface F^{n-1} is a hyperplane of the third kind if and only if the hypersurface F^{*n-1} is a hyperplane of the third kind.*

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