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Hilbert Flow Spaces with Operators over Topological Graphs

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Abstract: A complex system $\mathscr{S}$ consists m components, maybe inconsistence with $m \geq 2$, such as those of biological systems or generally, interaction systems and usually, a system with contradictions, which implies that there are no a mathematical subfield applicable. Then, *how can we hold on its global and local behaviors or reality*? All of us know that there always exists universal connections between things in the world, i.e., a topological graph $\overrightarrow{G}$ underlying components in $\mathscr{S}$. We can thereby establish mathematics over graphs $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots$ by viewing labeling graphs $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots$ to be globally mathematical elements, not only game objects or combinatorial structures, which can be applied to characterize dynamic behaviors of the system $\mathscr{S}$ on time t. Formally, a continuity flow $\overrightarrow{G}^L$ is a topological graph $\overrightarrow{G}$ associated with a mapping $L : (v, u) \to L(v, u)$, 2 end-operators $A_{vu}^+ : L(v, u) \to L^{A_{vu}^+}(v, u)$ and $A_{uv}^+ : L(u, v) \to L^{A_{uv}^+}(u, v)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$ with $L(v, u) = -L(u, v)$ and $A_{vu}^+(-L(v, u)) = -L^{A_{vu}^+}(v, u)$ for $\forall (v, u) \in E\left(\overrightarrow{G}\right)$ holding with continuity equations

$$\sum_{u \in N_G(v)} L^{A_{vu}^+}(v, u) = L(v), \qquad \forall v \in V\left(\overrightarrow{G}\right).$$

The main purpose of this paper is to extend Banach or Hilbert spaces to Banach or Hilbert continuity flow spaces over topological graphs $\left\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots\right\}$ and establish differentials on continuity flows for characterizing their globally change rate. A few well-known results such as those of Taylor formula, L'Hospital's rule on limitation are generalized to continuity flows, and algebraic or differential flow equations are discussed in this paper. All of these results form the elementary differential theory on continuity flows, which contributes mathematical combinatorics and can be used to characterizing the behavior of complex systems, particularly, the synchronization.

Key Words: Complex system, Smarandache multispace, continuity flow, Banach space, Hilbert space, differential, Taylor formula, L'Hospital's rule, mathematical combinatorics.

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§1. Introduction

A *Banach* or *Hilbert space* is respectively a linear space $\mathscr{A}$ over a field $\mathbb{R}$ or $\mathbb{C}$ equipped with a complete norm $\|\cdot\|$ or inner product $\langle \cdot , \cdot \rangle$, i.e., for every Cauchy sequence $\{x_n\}$ in $\mathscr{A}$, there

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exists an element x in $\mathscr{A}$ such that

$$\lim_{n\to\infty}\|x_n - x\|_{\mathscr{A}} = 0 \quad \text{or} \quad \lim_{n\to\infty}\langle x_n - x, x_n - x\rangle_{\mathscr{A}} = 0$$

and a topological graph $\varphi(G)$ is an embedding of a graph G with vertex set $V(G)$, edge set $E(G)$ in a space $\mathscr{S}$, i.e., there is a $1-1$ continuous mapping $\varphi: G \to \varphi(G) \subset \mathscr{S}$ with $\varphi(p) \neq \varphi(q)$ if $p \neq q$ for $\forall p, q \in G$, i.e., edges of G only intersect at vertices in $\mathscr{S}$, an embedding of a topological space to another space. A well-known result on embedding of graphs without loops and multiple edges in $\mathbb{R}^n$ concluded that *there always exists an embedding of G that all edges are straight segments in $\mathbb{R}^n$ for $n > 3$* ([22]) such as those shown in Fig.1.

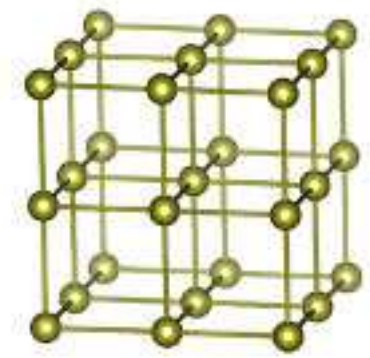

Fig.1

As we known, the purpose of science is hold on the reality of things in the world. However, the reality of a thing $\mathscr{T}$ is complex and there are no a mathematical subfield applicable unless a system maybe with contradictions in general. *Is such a contradictory system meaningless to human beings*? Certain not because all of these contradictions are the result of human beings, not the nature of things themselves, particularly on those of contradictory systems in mathematics. Thus, holding on the reality of things motivates one to turn contradictory systems to compatible one by a combinatorial notion and establish an envelope theory on mathematics, i.e., mathematical combinatorics ([9]-[13]). Then, *Can we globally characterize the behavior of a system or a population with elements≥ 2, which maybe contradictory or compatible*? The answer is certainly YES by *continuity flows*, which needs one to establish an envelope mathematical theory over topological graphs, i.e., views labeling graphs G^L to be mathematical elements ([19]), not only a game object or a combinatorial structure with labels in the following sense.

Definition 1.1 *A continuity flow $\left(\overrightarrow{G}; L, A\right)$ is an oriented embedded graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L: v \to L(v)$, $(v,u) \to L(v,u)$, 2 end-operators A^+_{vu} : $L(v,u) \to L^{A^+_{vu}}(v,u)$ and $A^+_{uv}: L(u,v) \to L^{A^+_{uv}}(u,v)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$*

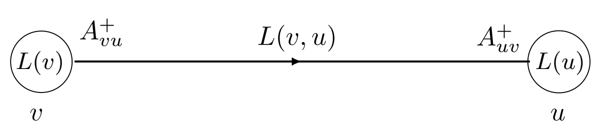

Fig.2

with $L(v,u) = -L(u,v)$ *and* $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ *holding with continuity equation*

$$\sum_{u \in N_G(v)} L^{A^+_{vu}}(v,u) = L(v) \quad for \quad \forall v \in V\left(\overrightarrow{G}\right)$$

such as those shown for vertex v *in Fig.*3 *following*

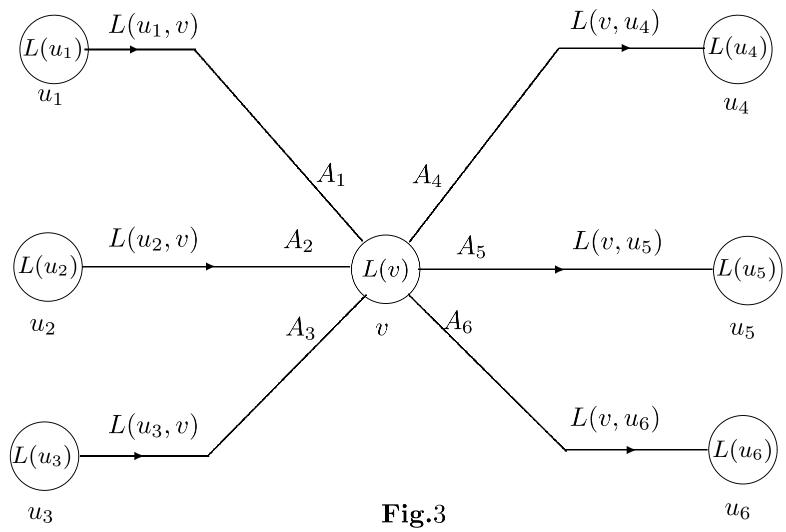

Fig.3

with a continuity equation

$$L^{A_1}(v,u_1) + L^{A_2}(v,u_2) + L^{A_3}(v,u_3) - L^{A_4}(v,u_4) - L^{A_5}(v,u_5) - L^{A_6}(v,u_6) = L(v),$$

where $L(v)$ *is the surplus flow on vertex* v.

Particularly, if $L(v) = \dot{x}_v$ *or constants* $\mathbf{v}_v, v \in V\left(\overrightarrow{G}\right)$, *the continuity flow* $\left(\overrightarrow{G}; L, A\right)$ *is respectively said to be a complex flow or an action* A *flow, and* $\overrightarrow{G}$*-flow if* $A = \mathbf{1}_{\mathscr{V}}$, *where* $\dot{x}_v = dx_v/dt$, x_v *is a variable on vertex* v *and* $\mathbf{v}$ *is an element in* $\mathscr{B}$ *for* $\forall v \in E\left(\overrightarrow{G}\right)$.

Clearly, an action flow is an equilibrium state of a continuity flow $\left(\overrightarrow{G}; L, A\right)$. We have shown that Banach or Hilbert space can be extended over topological graphs ([14],[17]), which can be applied to understanding the reality of things in [15]-[16], and we also shown that complex flows can be applied to hold on the global stability of biological n-system with $n \geq 3$ in [19]. For further discussing continuity flows, we need conceptions following.

Definition 1.2 *Let* $\mathscr{B}_1, \mathscr{B}_2$ *be Banach spaces over a field* $\mathbb{F}$ *with norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$, *respectively. An operator* $\mathbf{T} : \mathscr{B}_1 \to \mathscr{B}_2$ *is linear if*

$$\mathbf{T}(\lambda \mathbf{v}_1 + \mu \mathbf{v}_2) = \lambda \mathbf{T}(\mathbf{v}_1) + \mu \mathbf{T}(\mathbf{v}_2)$$

for $\lambda, \mu \in \mathbb{F}$, *and* $\mathbf{T}$ *is said to be continuous at a vector* $\mathbf{v}_0$ *if there always exist such a number*

$\delta(\varepsilon)$ for $\forall\epsilon > 0$ that

$$\left\|\mathbf{T}\left(\mathbf{v}\right)-\mathbf{T}\left(\mathbf{v}_0\right)\right\|_2 < \varepsilon$$

if $\left\|\mathbf{v}-\mathbf{v}_0\right\|_1 < \delta(\varepsilon)$ for $\forall\mathbf{v},\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2 \in \mathscr{B}_1$.

Definition 1.3 *Let $\mathscr{B}_1,\mathscr{B}_2$ be Banach spaces over a field $\mathbb{F}$ with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. An operator $\mathbf{T}:\mathscr{B}_1\rightarrow\mathscr{B}_2$ is bounded if there is a constant $M>0$ such that*

$$\left\|\mathbf{T}\left(\mathbf{v}\right)\right\|_2 \leq M\left\|\mathbf{v}\right\|_1, \quad i.e., \quad \frac{\|\mathbf{T}(\mathbf{v})\|_2}{\|\mathbf{v}\|_1} \leq M$$

for $\forall\mathbf{v}\in\mathscr{B}$ and furthermore, $\mathbf{T}$ is said to be a contractor if

$$\left\|\mathbf{T}\left(\mathbf{v}_1\right)-\mathbf{T}\left(\mathbf{v}_2\right)\right\| \leq c\left\|\mathbf{v}_1-\mathbf{v}_2)\right\|$$

for $\forall\mathbf{v}_1,\mathbf{v}_2\in\mathscr{B}$ with $c\in[0,1)$.

We only discuss the case that all end-operators A^+_{vu}, A^+_{uv} are both linear and continuous. In this case, the result following on linear operators of Banach space is useful.

Theorem 1.4 *Let $\mathscr{B}_1,\mathscr{B}_2$ be Banach spaces over a field $\mathbb{F}$ with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then, a linear operator $\mathbf{T}:\mathscr{B}_1\rightarrow\mathscr{B}_2$ is continuous if and only if it is bounded, or equivalently,*

$$\|\mathbf{T}\| := \sup_{\mathbf{0}\neq\mathbf{v}\in\mathscr{B}_1} \frac{\|\mathbf{T}(\mathbf{v})\|_2}{\|\mathbf{v}\|_1} < +\infty.$$

Let $\left\{\overrightarrow{G}_1,\overrightarrow{G}_2,\cdots\right\}$ be a graph family. The main purpose of this paper is to extend Banach or Hilbert spaces to Banach or Hilbert continuity flow spaces over topological graphs $\left\{\overrightarrow{G}_1,\overrightarrow{G}_2,\cdots\right\}$ and establish differentials on continuity flows, which enables one to characterize their globally change rate constraint on the combinatorial structure. A few well-known results such as those of Taylor formula, L'Hospital's rule on limitation are generalized to continuity flows, and algebraic or differential flow equations are discussed in this paper. All of these results form the elementary differential theory on continuity flows, which contributes mathematical combinatorics and can be used to characterizing the behavior of complex systems, particularly, the synchronization.

For terminologies and notations not defined in this paper, we follow references [1] for mechanics, [4] for functionals and linear operators, [22] for topology, [8] combinatorial geometry, [6]-[7],[25] for Smarandache systems, Smarandache geometries and Smaarandache multispaces and [2], [20] for biological mathematics.

§2. Banach and Hilbert Flow Spaces

2.1 Linear Spaces over Graphs

Let $\overrightarrow{G}_1,\overrightarrow{G}_2,\cdots,\overrightarrow{G}_n$ be oriented graphs embedded in topological space $\mathscr{S}$ with $\overrightarrow{\mathscr{G}}=\bigcup\limits_{i=1}^{n}\overrightarrow{G}_i$, i.e., $\overrightarrow{G}_i$ is a subgraph of $\overrightarrow{\mathscr{G}}$ for integers $1 \le i \le n$. In this case, these is naturally an embedding $\iota : \overrightarrow{G}_i \to \overrightarrow{\mathscr{G}}$.

Let $\mathscr{V}$ be a linear space over a field $\mathscr{F}$. A vector labeling $L : \overrightarrow{G} \to \mathscr{V}$ is a mapping with $L(v), L(e) \in \mathscr{V}$ for $\forall v \in V(\overrightarrow{G}), e \in E(\overrightarrow{G})$. Define

$$\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} = \left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)^{L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_1+L_2} \bigcup \left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)^{L_2} \tag{2.1}$$

and

$$\lambda \cdot \overrightarrow{G}^L = \overrightarrow{G}^{\lambda \cdot L} \tag{2.2}$$

for $\forall \lambda \in \mathscr{F}$. Clearly, if , and $\overrightarrow{G}^L, \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}$ are continuity flows with linear end-operators A_{vu}^+ and A_{uv}^+ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}$ and $\lambda \cdot \overrightarrow{G}^L$ are continuity flows also. If we consider each continuity flow $\overrightarrow{G}_i^L$ a continuity subflow of $\overrightarrow{\mathscr{G}}^{\widehat{L}}$, where $\widehat{L} : \overrightarrow{G}_i = L(\overrightarrow{G}_i)$ but $\widehat{L} : \overrightarrow{\mathscr{G}} \setminus \overrightarrow{G}_i \to \mathbf{0}$ for integers $1 \le i \le n$, and define $\mathbf{O} : \overrightarrow{\mathscr{G}} \to \mathbf{0}$, then all continuity flows, particularly, all complex flows, or all action flows on oriented graphs $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ naturally form a linear space, denoted by $\left(\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}; +, \cdot\right)$ over a field $\mathscr{F}$ under operations (2.1) and (2.2) because it holds with:

(1) A field $\mathscr{F}$ of scalars;

(2) A set $\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}$ of objects, called continuity flows;

(3) An operation "+", called continuity flow addition, which associates with each pair of continuity flows $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}$ in $\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}$ a continuity flows $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}$ in $\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}$, called the sum of $\overrightarrow{G}_1^{L_1}$ and $\overrightarrow{G}_2^{L_2}$, in such a way that

(a) Addition is commutative, $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} = \overrightarrow{G}_2^{L_2} + \overrightarrow{G}_1^{L_1}$ because of

$$\begin{aligned} \overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} &= \left(\overrightarrow{G}_1 - \overrightarrow{G}_2\right)^{L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_1+L_2} \bigcup \left(\overrightarrow{G}_2 - \overrightarrow{G}_1\right)^{L_2} \\ &= \left(\overrightarrow{G}_2 - \overrightarrow{G}_1\right)^{L_2} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_2+L_1} \bigcup \left(\overrightarrow{G}_1 - \overrightarrow{G}_2\right)^{L_1} \\ &= \overrightarrow{G}_2^{L_2} + \overrightarrow{G}_1^{L_1}; \end{aligned}$$

(b) Addition is associative, $\left(\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right) + \overrightarrow{G}_3^{L_3} = \overrightarrow{G}_1^{L_1} + \left(\overrightarrow{G}_2^{L_2} + \overrightarrow{G}_3^{L_3}\right)$ because if we let

$$L_{ijk}^+(x) = \begin{cases} L_i(x), & \text{if } x \in \overrightarrow{G}_i \setminus \left(\overrightarrow{G}_j \bigcup \overrightarrow{G}_k\right) \\ L_j(x), & \text{if } x \in \overrightarrow{G}_j \setminus \left(\overrightarrow{G}_i \bigcup \overrightarrow{G}_k\right) \\ L_k(x), & \text{if } x \in \overrightarrow{G}_k \setminus \left(\overrightarrow{G}_i \bigcup \overrightarrow{G}_j\right) \\ L_{ij}^+(x), & \text{if } x \in \left(\overrightarrow{G}_i \bigcap \overrightarrow{G}_j\right) \setminus \overrightarrow{G}_k \\ L_{ik}^+(x), & \text{if } x \in \left(\overrightarrow{G}_i \bigcap \overrightarrow{G}_k\right) \setminus \overrightarrow{G}_j \\ L_{jk}^+(x), & \text{if } x \in \left(\overrightarrow{G}_j \bigcap \overrightarrow{G}_k\right) \setminus \overrightarrow{G}_i \\ L_i(x) + L_j(x) + L_k(x) & \text{if } x \in \overrightarrow{G}_i \bigcap \overrightarrow{G}_j \bigcap \overrightarrow{G}_k \end{cases} \tag{2.3}$$

and

$$L_{ij}^{+}(x)=\begin{cases} L_i(x), & \text{if } x\in\overrightarrow{G}_i\setminus\overrightarrow{G}_j\\ L_j(x), & \text{if } x\in\overrightarrow{G}_j\setminus\overrightarrow{G}_i\\ L_i(x)+L_j(x), & \text{if } x\in\overrightarrow{G}_i\bigcap\overrightarrow{G}_j\end{cases} \tag{2.4}$$

for integers $1\le i,j,k\le n$, then

$$\begin{aligned}\left(\overrightarrow{G}_1^{L_1}+\overrightarrow{G}_2^{L_2}\right)+\overrightarrow{G}_3^{L_3} &= \left(\overrightarrow{G}_1\bigcup\overrightarrow{G}_2\right)^{L_{12}^{+}}+\overrightarrow{G}_3^{L_3}=\left(\overrightarrow{G}_1\bigcup\overrightarrow{G}_2\bigcup\overrightarrow{G}_3\right)^{L_{123}^{+}}\\ &= \overrightarrow{G}_1^{L_1}+\left(\overrightarrow{G}_2\bigcup\overrightarrow{G}_3\right)^{L_{23}^{+}}=\overrightarrow{G}_1^{L_1}+\left(\overrightarrow{G}_2^{L_2}+\overrightarrow{G}_3^{L_3}\right);\end{aligned}$$

(c) There is a unique continuity flow $\mathbf{O}$ on $\overrightarrow{\mathscr{G}}$ hold with $\mathbf{O}(v,u)=\mathbf{0}$ for $\forall(v,u)\in E\left(\overrightarrow{\mathscr{G}}\right)$ and $V\left(\overrightarrow{\mathscr{G}}\right)$ in $\left\langle\overrightarrow{G}_i,1\le i\le n\right\rangle^{\mathscr{V}}$, called zero such that $\overrightarrow{G}^L+\mathbf{O}=\overrightarrow{G}^L$ for $\overrightarrow{G}^L\in\left\langle\overrightarrow{G}_i,1\le i\le n\right\rangle^{\mathscr{V}}$;

(d) For each continuity flow $\overrightarrow{G}^L\in\left\langle\overrightarrow{G}_i,1\le i\le n\right\rangle^{\mathscr{V}}$ there is a unique continuity flow $\overrightarrow{G}^{-L}$ such that $\overrightarrow{G}^L+\overrightarrow{G}^{-L}=\mathbf{O}$;

(4) An operation "$\cdot$, called scalar multiplication, which associates with each scalar k in F and a continuity flow $\overrightarrow{G}^L$ in $\left\langle\overrightarrow{G}_i,1\le i\le n\right\rangle^{\mathscr{V}}$ a continuity flow $k\cdot\overrightarrow{G}^L$ in $\mathscr{V}$, called the product of k with $\overrightarrow{G}^L$, in such a way that

(a) $1\cdot\overrightarrow{G}^L=\overrightarrow{G}^L$ for every $\overrightarrow{G}^L$ in $\left\langle\overrightarrow{G}_i,1\le i\le n\right\rangle^{\mathscr{V}}$;

(b) $(k_1k_2)\cdot\overrightarrow{G}^L=k_1(k_2\cdot\overrightarrow{G}^L)$;

(c) $k\cdot(\overrightarrow{G}_1^{L_1}+\overrightarrow{G}_2^{L_2})=k\cdot\overrightarrow{G}_1^{L_1}+k\cdot\overrightarrow{G}_2^{L_2}$;

(d) $(k_1+k_2)\cdot\overrightarrow{G}^L=k_1\cdot\overrightarrow{G}^L+k_2\cdot\overrightarrow{G}^L$.

Usually, we abbreviate $\left(\left\langle\overrightarrow{G}_i,1\le i\le n\right\rangle^{\mathscr{V}};+,\cdot\right)$ to $\left\langle\overrightarrow{G}_i,1\le i\le n\right\rangle^{\mathscr{V}}$ if these operations $+$ and $\cdot$ are clear in the context.

By operation (1.1), $\overrightarrow{G}_1^{L_1}+\overrightarrow{G}_2^{L_2}\ne\overrightarrow{G}_1^{L_1}$ if and only if $\overrightarrow{G}_1\not\succeq\overrightarrow{G}_2$ with $L_1:\overrightarrow{G}_1\setminus\overrightarrow{G}_2\not\to\mathbf{0}$ and $\overrightarrow{G}_1^{L_1}+\overrightarrow{G}_2^{L_2}\ne\overrightarrow{G}_2^{L_2}$ if and only if $\overrightarrow{G}_2\not\succeq\overrightarrow{G}_1$ with $L_2:\overrightarrow{G}_2\setminus\overrightarrow{G}_1\not\to\mathbf{0}$, which allows us to introduce the conception of linear irreducible. Generally, a continuity flow family $\{\overrightarrow{G}_1^{L_1},\overrightarrow{G}_2^{L_2},\cdots,\overrightarrow{G}_n^{L_n}\}$ is *linear irreducible* if for any integer i,

$$\overrightarrow{G}_i\not\succeq\bigcup_{l\ne i}\overrightarrow{G}_l\quad\text{with}\quad L_i:\overrightarrow{G}_i\setminus\bigcup_{l\ne i}\overrightarrow{G}_l\not\to\mathbf{0}, \tag{2.5}$$

where $1\le i\le n$. We know the following result on linear generated sets.

Theorem 2.1 *Let $\mathscr{V}$ be a linear space over a field $\mathscr{F}$ and let $\left\{\overrightarrow{G}_1^{L_1},\overrightarrow{G}_2^{L_2},\cdots,\overrightarrow{G}_n^{L_n}\right\}$ be an linear irreducible family, $L_i:\overrightarrow{G}_i\to\mathscr{V}$ for integers $1\le i\le n$ with linear operators A_{vu}^{+}, A_{uv}^{+} for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. Then, $\left\{\overrightarrow{G}_1^{L_1},\overrightarrow{G}_2^{L_2},\cdots,\overrightarrow{G}_n^{L_n}\right\}$ is an independent generated set of*

$\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$, *called basis, i.e.,*

$$\dim \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}} = n.$$

Proof By definition, $\overrightarrow{G}_i^{L_i}, 1 \leq i \leq n$ is a linear generated of $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ with $L_i : \overrightarrow{G}_i \to \mathscr{V}$, i.e.,

$$\dim \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}} \leq n.$$

We only need to show that $\overrightarrow{G}_i^{L_i}, 1 \leq i \leq n$ is linear independent, i.e.,

$$\dim \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}} \geq n,$$

which implies that if there are n scalars $c_1, c_2, \cdots, c_n$ holding with

$$c_1 \overrightarrow{G}_1^{L_1} + c_2 \overrightarrow{G}_2^{L_2} + \cdots + c_n \overrightarrow{G}_n^{L_n} = \mathbf{O},$$

then $c_1 = c_2 = \cdots = c_n = 0$. Notice that $\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n\}$ is linear irreducible. We are easily know $\overrightarrow{G_i} \setminus \bigcup\limits_{l \neq i} \overrightarrow{G}_l \neq \emptyset$ and find an element $x \in E(\overrightarrow{G_i} \setminus \bigcup\limits_{l \neq i} \overrightarrow{G}_l)$ such that $c_i L_i(x) = \mathbf{0}$ for integer $i, 1 \leq i \leq n$. However, $L_i(x) \neq \mathbf{0}$ by (1.5). We get that $c_i = 0$ for integers $1 \leq i \leq n$. □

A subspace of $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ is called an A_0-*flow space* if its elements are all continuity flows $\overrightarrow{G}^L$ with $L(v)$, $v \in V\left(\overrightarrow{G}\right)$ are constant $\mathbf{v}$. The result following is an immediately conclusion of Theorem 2.1.

Theorem 2.2 *Let $\overrightarrow{G}, \overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ be oriented graphs embedded in a space $\mathscr{S}$ and $\mathscr{V}$ a linear space over a field $\mathscr{F}$. If $\overrightarrow{G}^{\mathbf{v}}, \overrightarrow{G}_1^{\mathbf{v}_1}, \overrightarrow{G}_2^{\mathbf{v}_2}, \cdots, \overrightarrow{G}_n^{\mathbf{v}_n}$ are continuity flows with $\mathbf{v}(v) = \mathbf{v}, \mathbf{v}_i(v) = \mathbf{v}_i \in \mathscr{V}$ for $\forall v \in V\left(\overrightarrow{G}\right)$, $1 \leq i \leq n$, then*

(1) $\left\langle \overrightarrow{G}^{\mathbf{v}} \right\rangle$ *is an A_0-flow space;*

(2) $\left\langle \overrightarrow{G}_1^{\mathbf{v}_1}, \overrightarrow{G}_2^{\mathbf{v}_2}, \cdots, \overrightarrow{G}_n^{\mathbf{v}_n} \right\rangle$ *is an A_0-flow space if and only if $\overrightarrow{G}_1 = \overrightarrow{G}_2 = \cdots = \overrightarrow{G}_n$ or* $\mathbf{v}_1 = \mathbf{v}_2 = \cdots = \mathbf{v}_n = \mathbf{0}$.

Proof By definition, $\overrightarrow{G}_1^{\mathbf{v}_1} + \overrightarrow{G}_2^{\mathbf{v}_2}$ and $\lambda \overrightarrow{G}^{\mathbf{v}}$ are A_0-flows if and only if $\overrightarrow{G}_1 = \overrightarrow{G}_1$ or $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}$ by definition. We therefore know this result. □

2.2 Commutative Rings over Graphs

Furthermore, if $\mathscr{V}$ is a commutative ring $(\mathscr{R}; +, \cdot)$, we can extend it over oriented graph family $\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n\}$ by introducing operation $+$ with (2.1) and operation $\cdot$ following:

$$\overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_2^{L_2} = \left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)^{L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_1 \cdot L_2} \bigcup \left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)^{L_2}, \quad (2.6)$$

where $L_1 \cdot L_2 : x \to L_1(x) \cdot L_2(x)$, and particularly, the scalar product for $\mathbb{R}^n, n \geq 2$ for $x \in \overrightarrow{G}_1 \bigcap \overrightarrow{G}_2$.

As we shown in Subsection 2.1, $\left(\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{R}}; +\right)$ is an Abelian group. We show $\left(\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{R}}; +, \cdot\right)$ is a commutative semigroup also.

In fact, define

$$L_{ij}^{\times}(x) = \begin{cases} L_i(x), & \text{if } x \in \overrightarrow{G}_i \setminus \overrightarrow{G}_j \\ L_j(x), & \text{if } x \in \overrightarrow{G}_j \setminus \overrightarrow{G}_i \\ L_i(x) \cdot L_j(x), & \text{if } x \in \overrightarrow{G}_i \bigcap \overrightarrow{G}_j \end{cases}$$

Then, we are easily known that $\overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_2^{L_2} = \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)^{L_{12}^{\times}} = \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)^{L_{21}^{\times}} = \overrightarrow{G}_2^{L_2} \cdot \overrightarrow{G}_1^{L_1}$ for $\forall \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \left(\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{R}}; \cdot\right)$ by definition (2.6), i.e., it is commutative.

Let

$$L_{ijk}^{\times}(x) = \begin{cases} L_i(x), & \text{if } x \in \overrightarrow{G}_i \setminus \left(\overrightarrow{G}_j \bigcup \overrightarrow{G}_k\right) \\ L_j(x), & \text{if } x \in \overrightarrow{G}_j \setminus \left(\overrightarrow{G}_i \bigcup \overrightarrow{G}_k\right) \\ L_k(x), & \text{if } x \in \overrightarrow{G}_k \setminus \left(\overrightarrow{G}_i \bigcup \overrightarrow{G}_j\right) \\ L_{ij}(x), & \text{if } x \in \left(\overrightarrow{G}_i \bigcap \overrightarrow{G}_j\right) \setminus \overrightarrow{G}_k \\ L_{ik}(x), & \text{if } x \in \left(\overrightarrow{G}_i \bigcap \overrightarrow{G}_k\right) \setminus \overrightarrow{G}_j \\ L_{jk}(x), & \text{if } x \in \left(\overrightarrow{G}_j \bigcap \overrightarrow{G}_k\right) \setminus \overrightarrow{G}_i \\ L_i(x) \cdot L_j(x) \cdot L_k(x) & \text{if } x \in \overrightarrow{G}_i \bigcap \overrightarrow{G}_j \bigcap \overrightarrow{G}_k \end{cases}$$

Then,

$$\begin{aligned} \left(\overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_2^{L_2}\right) \cdot \overrightarrow{G}_3^{L_3} &= \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)^{L_{12}^{\times}} \cdot \overrightarrow{G}_3^{L_3} = \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2 \bigcup \overrightarrow{G}_3\right)^{L_{123}^{\times}} \\ &= \overrightarrow{G}_1^{L_1} \cdot \left(\overrightarrow{G}_2 \bigcup \overrightarrow{G}_3\right)^{L_{23}^{\times}} = \overrightarrow{G}_1^{L_1} \cdot \left(\overrightarrow{G}_2^{L_2} \cdot \overrightarrow{G}_3^{L_3}\right). \end{aligned}$$

Thus,

$$\left(\overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_2^{L_2}\right) \cdot \overrightarrow{G}_3^{L_3} = \overrightarrow{G}_1^{L_1} \cdot \left(\overrightarrow{G}_2^{L_2} \cdot \overrightarrow{G}_3^{L_3}\right)$$

for $\forall \overrightarrow{G}^{L}, \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \left(\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{R}}; \cdot\right)$, which implies that it is a semigroup.

We are also need to verify the distributive laws, i.e.,

$$\overrightarrow{G}_3^{L_3} \cdot \left(\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right) = \overrightarrow{G}_3^{L_3} \cdot \overrightarrow{G}_1^{L_1} + \overrightarrow{G}_3^{L_3} \cdot \overrightarrow{G}_2^{L_2} \tag{2.7}$$

and

$$\left(\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right) \cdot \overrightarrow{G}_3^{L_3} = \overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_3^{L_3} + \overrightarrow{G}_2^{L_2} \cdot \overrightarrow{G}_3^{L_3} \tag{2.8}$$

for $\forall\overrightarrow{G}_3, \overrightarrow{G}_1, \overrightarrow{G}_2 \in \left(\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathscr{R}}; +, \cdot\right)$. Clearly,

$$\begin{aligned}\overrightarrow{G}_3^{L_3}\cdot\left(\overrightarrow{G}_1^{L_1}+\overrightarrow{G}_2^{L_2}\right) &= \overrightarrow{G}_3^{L_3}\cdot\left(\overrightarrow{G}_1\bigcup\overrightarrow{G}_2\right)^{L_{12}^{+}} = \left(\overrightarrow{G}_3\left(\overrightarrow{G}_1\bigcup\overrightarrow{G}_2\right)\right)^{L_{3(21)}^{\times}}\\ &= \left(\overrightarrow{G}_3\bigcup\overrightarrow{G}_1\right)^{L_{31}^{\times}}\bigcup\left(\overrightarrow{G}_3\bigcup\overrightarrow{G}_2\right)^{L_{32}^{\times}} = \overrightarrow{G}_3^{L_3}\cdot\overrightarrow{G}_1^{L_1}+\overrightarrow{G}_3^{L_3}\cdot\overrightarrow{G}_2^{L_2},\end{aligned}$$

which is the (2.7). The proof for (2.8) is similar. Thus, we get the following result.

Theorem 2.3 *Let* $(\mathscr{R};+,\cdot)$ *be a commutative ring and let* $\left\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\right\}$ *be a linear irreducible family,* $L_i : \overrightarrow{G}_i \to \mathscr{R}$ *for integers* $1\leq i\leq n$ *with linear operators* A_{vu}^{+}, A_{uv}^{+} *for* $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. *Then,* $\left(\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathscr{R}}; +, \cdot\right)$ *is a commutative ring.*

2.3 Banach or Hilbert Flow Spaces

Let $\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\}$ be a basis of $\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathscr{V}}$, where $\mathscr{V}$ is a Banach space with a norm $\|\cdot\|$. For $\forall\overrightarrow{G}^L \in \left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathscr{V}}$, define

$$\left\|\overrightarrow{G}^L\right\| = \sum_{e\in E\left(\overrightarrow{G}\right)} \|L(e)\|. \tag{2.9}$$

Then, for $\forall\overrightarrow{G}, \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathscr{V}}$ we are easily know that

(1) $\left\|\overrightarrow{G}^L\right\|\geq 0$ and $\left\|\overrightarrow{G}^L\right\| = 0$ if and only if $\overrightarrow{G}^L = \mathbf{O}$;

(2) $\left\|\overrightarrow{G}^{\xi L}\right\| = \xi\left\|\overrightarrow{G}^L\right\|$ for any scalar ξ;

(3) $\left\|\overrightarrow{G}_1^{L_1}+\overrightarrow{G}_2^{L_2}\right\| \leq \left\|\overrightarrow{G}_1^{L_1}\right\| + \left\|\overrightarrow{G}_2^{L_2}\right\|$ because of

$$\begin{aligned}\left\|\overrightarrow{G}_1^{L_1}+\overrightarrow{G}_2^{L_2}\right\| &= \sum_{e\in E\left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)}\|L_1(e)\|\\ &\quad+\sum_{e\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)}\|L_1(e)+L_2(e)\| + \sum_{e\in E\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)}\|L_2(e)\|\\ &\leq \left(\sum_{e\in E\left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)}\|L_1(e)\| + \sum_{e\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)}\|L_1(e)\|\right)\\ &\quad+\left(\sum_{e\in E\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)}\|L_2(e)\| + \sum_{e\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)}\|L_2(e)\|\right) = \left\|\overrightarrow{G}_1^{L_1}\right\| + \left\|\overrightarrow{G}_2^{L_2}\right\|.\end{aligned}$$

for $\|L_1(e)+L_2(e)\|\leq\|L_1(e)\|+\|L_2(e)\|$ in Banach space $\mathscr{V}$. Therefore, $\|\cdot\|$ is also a norm on $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$.

Furthermore, if $\mathscr{V}$ is a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, for $\forall \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$, define

$$\begin{aligned} \left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \right\rangle &= \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)} \langle L_1(e), L_1(e) \rangle \\ &+ \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)} \langle L_1(e), L_2(e) \rangle + \sum_{e \in E\left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)} \langle L_2(e), L_2(e) \rangle . \end{aligned} \tag{2.10}$$

Then we are easily know also that

(1) For $\forall \overrightarrow{G}^L \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$,

$$\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle = \sum_{e \in E\left(\overrightarrow{G}\right)} \langle L(e), L(e) \rangle \geq 0$$

and $\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle = 0$ if and only if $\overrightarrow{G}^L = \mathbf{O}$.

(2) For $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$,

$$\left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \right\rangle = \overline{\left\langle \overrightarrow{G}_2^{L_2}, \overrightarrow{G}_1^{L_1} \right\rangle}$$

because of

$$\begin{aligned} \left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \right\rangle &= \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)} \langle L_1(e), L_1(e) \rangle + \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)} \langle L_1(e), L_2(e) \rangle \\ &+ \sum_{e \in E\left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)} \langle L_2(e), L_2(e) \rangle \\ &= \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)} \overline{\langle L_1(e), L_1(e) \rangle} + \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)} \overline{\langle L_2(e), L_1(e) \rangle} \\ &+ \sum_{e \in E\left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)} \overline{\langle L_2(e), L_2(e) \rangle} = \overline{\left\langle \overrightarrow{G}_2^{L_2}, \overrightarrow{G}_1^{L_1} \right\rangle} \end{aligned}$$

for $\langle L_1(e), L_2(e) \rangle = \overline{\langle L_2(e), L_1(e) \rangle}$ in Hilbert space $\mathscr{V}$.

(3) For $\overrightarrow{G}^L, \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ and $\lambda, \mu \in \mathscr{F}$, there is

$$\left\langle \lambda \overrightarrow{G}_1^{L_1} + \mu \overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L \right\rangle = \lambda \left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}^L \right\rangle + \mu \left\langle \overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L \right\rangle$$

because of

$$\left\langle \lambda\overrightarrow{G}_1^{L_1} + \mu\overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L \right\rangle = \left\langle \overrightarrow{G}_1^{\lambda L_1} + \overrightarrow{G}_2^{\mu L_2}, \overrightarrow{G}^L \right\rangle$$
$$= \left\langle \left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)^{\lambda L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{\lambda L_1 + \mu L_2} \bigcup \left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)^{\mu L_2}, \overrightarrow{G}^L \right\rangle.$$

Define $L_{1_\lambda 2_\mu} : \overrightarrow{G}_1 \bigcup \overrightarrow{G}_2 \to \mathscr{V}$ by

$$L_{1_\lambda 2_\mu}(x) = \begin{cases} \lambda L_1(x), & \text{if } x \in \overrightarrow{G}_1 \setminus \overrightarrow{G}_2 \\ \mu L_2(x), & \text{if } x \in \overrightarrow{G}_2 \setminus \overrightarrow{G}_1 \\ \lambda L_1(x) + \mu L_2(x), & \text{if } x \in \overrightarrow{G}_2 \bigcap \overrightarrow{G}_1 \end{cases}$$

Then, we know that

$$\begin{aligned} \left\langle \lambda\overrightarrow{G}_1^{L_1} + \mu\overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L \right\rangle &= \sum_{e \in E\left(\left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right) \setminus \overrightarrow{G}\right)} \left\langle L_{1_\lambda 2_\mu}(e), L_{1_\lambda 2_\mu}(e) \right\rangle \\ &+ \sum_{e \in E\left(\left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right) \bigcap \overrightarrow{G}\right)} \left\langle L_{1_\lambda 2_\mu}(e), L(e) \right\rangle \\ &+ \sum_{e \in E\left(\overrightarrow{G} \setminus \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)\right)} \langle L(e), L(e) \rangle. \end{aligned}$$

and

$$\begin{aligned} &\lambda \left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}^L \right\rangle + \mu \left\langle \overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L \right\rangle \\ &= \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)} \langle \lambda L_1(e), \lambda L_1(e) \rangle + \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)} \langle \lambda L_1(e), L(e) \rangle \\ &\quad + \sum_{e \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)} \langle L(e), L(e) \rangle + \sum_{e \in E\left(\overrightarrow{G}_2 \setminus \overrightarrow{G}\right)} \langle \mu L_2(e), \mu L_2(e) \rangle \\ &\quad + \sum_{e \in E\left(\overrightarrow{G}_2 \bigcap \overrightarrow{G}\right)} \langle \mu L_2(e), L(e) \rangle + \sum_{e \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_2\right)} \langle L(e), L(e) \rangle. \end{aligned}$$

Notice that

$$\begin{aligned} &\sum_{e \in E\left(\left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right) \setminus \overrightarrow{G}\right)} \left\langle L_{1_\lambda 2_\mu}(e), L_{1_\lambda 2_\mu}(e) \right\rangle \\ &= \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)} \langle \lambda L_1(e), \lambda L_1(e) \rangle + \sum_{e \in E\left(\overrightarrow{G}_2 \setminus \overrightarrow{G}\right)} \langle \mu L_2(e), \mu L_2(e) \rangle \\ &\quad + \sum_{e \in E\left(\left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right) \bigcap \overrightarrow{G}\right)} \left\langle L_{1_\lambda 2_\mu}(e), L(e) \right\rangle \end{aligned}$$

$$= \sum_{e\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}\right)} \langle\lambda L_1(e), L(e)\rangle + \sum_{e\in E\left(\overrightarrow{G}_2\bigcap\overrightarrow{G}\right)} \langle\mu L_2(e), L(e)\rangle$$
$$+ \sum_{e\in E\left(\overrightarrow{G}\setminus\overrightarrow{G}_2\right)} \langle L(e), L(e)\rangle$$
$$= \sum_{e\in E\left(\overrightarrow{G}\setminus\overrightarrow{G}_1\right)} \langle L(e), L(e)\rangle + \sum_{e\in E\left(\overrightarrow{G}\setminus\overrightarrow{G}_2\right)} \langle L(e), L(e)\rangle .$$

We therefore know that

$$\left\langle\lambda\overrightarrow{G}_1^{L_1} + \mu\overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L\right\rangle = \lambda\left\langle\overrightarrow{G}_1^{L_1}, \overrightarrow{G}^L\right\rangle + \mu\left\langle\overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L\right\rangle.$$

Thus, $\overrightarrow{G}^{\mathscr{V}}$ is an inner space

If $\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\}$ is a basis of space $\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathscr{V}}$, we are easily find the exact formula on L by $L_1.L_2,\cdots,L_n$. In fact, let

$$\overrightarrow{G}^L = \lambda_1\overrightarrow{G}_1^{L_1} + \lambda_2\overrightarrow{G}_2^{L_2} + \cdots + \lambda_n\overrightarrow{G}_n^{L_n},$$

where $(\lambda_1, \lambda_2, \cdots, \lambda_n) \neq (0, 0, \cdots, 0)$, and let

$$\widehat{L} : \left(\bigcap_{l=1}^{i}\overrightarrow{G}_{k_l}\right) \setminus \left(\bigcup_{s\neq k_l,\cdots,k_i}\overrightarrow{G}_s\right) \to \sum_{l=1}^{i}\lambda_{k_l}L_{k_l}$$

for integers $1\leq i\leq n$. Then, we are easily knowing that $\widehat{L}$ is nothing else but the labeling L on $\overrightarrow{G}$ by operation (2.1), a generation of (2.3) and (2.4) with

$$\left\|\overrightarrow{G}^L\right\| = \sum_{i=1}^{n}\sum_{e\in E\left(\overrightarrow{G}_i\right)}\left\|\sum_{l=1}^{i}\lambda_{k_l}L_{k_l}(e)\right\|, \tag{2.11}$$

$$\left\langle\overrightarrow{G}^L, \overrightarrow{G}'^{L'}\right\rangle = \sum_{i=1}^{n}\sum_{e\in E\left(\overrightarrow{G}_i\right)}\left\langle\sum_{l=1}^{i}\lambda_{k_l}L^1_{k_l}(e), \sum_{s=1}^{i}\lambda'_{k_s}L^2_{k_s}\right\rangle, \tag{2.12}$$

where $\overrightarrow{G}'^{L'} = \lambda'_1\overrightarrow{G}_1^{L_1} + \lambda'_2\overrightarrow{G}_2^{L_2} + \cdots + \lambda'_n\overrightarrow{G}_n^{L_n}$ and $\overrightarrow{G}_i = \left(\bigcap_{l=1}^{i}\overrightarrow{G}_{k_l}\right) \setminus \left(\bigcup_{s\neq k_l,\cdots,k_i}\overrightarrow{G}_s\right)$.

We therefore extend the Banach or Hilbert space $\mathscr{V}$ over graphs $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ following.

Theorem 2.4 *Let $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ be oriented graphs embedded in a space $\mathscr{S}$ and $\mathscr{V}$ a Banach space over a field $\mathscr{F}$. Then $\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathscr{V}}$ with linear operators A^+_{vu}, A^+_{uv} for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ is a Banach space, and furthermore, if $\mathscr{V}$ is a Hilbert space, $\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathscr{V}}$ is a Hilbert space too.*

Proof We have shown, $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ is a linear normed space or inner space if $\mathscr{V}$ is a linear normed space or inner space, and for the later, let

$$\left\| \overrightarrow{G}^L \right\| = \sqrt{\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle}$$

for $\overrightarrow{G}^L \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$. Then $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ is a normed space and furthermore, it is a Hilbert space if it is complete. Thus, we are only need to show that any Cauchy sequence is converges in $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$.

In fact, let $\left\{ \overrightarrow{H}_n^{L_n} \right\}$ be a Cauchy sequence in $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$, i.e., for any number $\varepsilon > 0$, there always exists an integer $N(\varepsilon)$ such that

$$\left\| \overrightarrow{H}_n^{L_n} - \overrightarrow{H}_m^{L_m} \right\| < \varepsilon$$

if $n, m \geq N(\varepsilon)$. Let $\mathscr{G}^{\mathscr{V}}$ be the continuity flow space on $\overrightarrow{\mathscr{G}} = \bigcup_{i=1}^{n} \overrightarrow{G}_i$. We embed each $\overrightarrow{H}_n^{L_n}$ to a $\overrightarrow{\mathscr{G}}^{\widehat{L}} \in \overrightarrow{\mathscr{G}}^{\mathscr{V}}$ by letting

$$\widehat{L}_n(e) = \begin{cases} L_n(e), & \text{if } \ e \in E\left(H_n\right) \\ \mathbf{0}, & \text{if } \ e \in E\left(\overrightarrow{\mathscr{G}} \setminus \overrightarrow{H}_n\right). \end{cases}$$

Then

$$\begin{aligned} \left\| \overrightarrow{\mathscr{G}}^{\widehat{L}_n} - \overrightarrow{\mathscr{G}}^{\widehat{L}_m} \right\| &= \sum_{e \in E\left(\overrightarrow{G}_n \setminus \overrightarrow{G}_m\right)} \left\| L_n(e) \right\| + \sum_{e \in E\left(\overrightarrow{G}_n \bigcap \overrightarrow{G}_m\right)} \left\| L_n(e) - L_m(e) \right\| \\ &+ \sum_{e \in E\left(\overrightarrow{G}_m \setminus \overrightarrow{G}_n\right)} \left\| -L_m(e) \right\| = \left\| \overrightarrow{H}_n^{L_n} - \overrightarrow{H}_m^{L_m} \right\| \leq \varepsilon. \end{aligned}$$

Thus, $\left\{ \overrightarrow{\mathscr{G}}^{\widehat{L}_n} \right\}$ is a Cauchy sequence also in $\overrightarrow{\mathscr{G}}^{\mathscr{V}}$. By definition,

$$\left\| \widehat{L}_n(e) - \widehat{L}_m(e) \right\| \leq \left\| \overrightarrow{\mathscr{G}}^{L_n} - \overrightarrow{\mathscr{G}}^{L_m} \right\| < \varepsilon,$$

i.e., $\{L_n(e)\}$ is a Cauchy sequence for $\forall e \in E\left(\overrightarrow{\mathscr{G}}\right)$, which is converges on in $\mathscr{V}$ by definition.

Let

$$\widehat{L}(e) = \lim_{n \to \infty} \widehat{L}_n(e)$$

for $\forall e \in E\left(\overrightarrow{\mathscr{G}}\right)$. Then it is clear that $\lim_{n \to \infty} \overrightarrow{\mathscr{G}}^{\widehat{L}_n} = \overrightarrow{\mathscr{G}}^{\widehat{L}}$, which implies that $\{\overrightarrow{\mathscr{G}}^{\widehat{L}_n}\}$, i.e., $\left\{ \overrightarrow{H}_n^{L_n} \right\}$ is converges to $\overrightarrow{\mathscr{G}}^{\widehat{L}} \in \overrightarrow{\mathscr{G}}^{\mathscr{V}}$, an element in $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ because of $\widehat{L}(e) \in \mathscr{V}$ for $\forall e \in E\left(\overrightarrow{\mathscr{G}}\right)$ and $\overrightarrow{\mathscr{G}} = \bigcup_{i=1}^{n} \overrightarrow{G}_i$. $\square$

§3. Differential on Continuity Flows

3.1 Continuity Flow Expansion

Theorem 2.4 enables one to establish differentials and generalizes results in classical calculus in space $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$. Let L be kth differentiable to t on a domain $\mathscr{D} \subset \mathbb{R}$, where $k \geq 1$. Define

$$\frac{d\overrightarrow{G}^L}{dt} = \overrightarrow{G}^{\frac{dL}{dt}} \quad \text{and} \quad \int_0^t \overrightarrow{G}^L dt = \overrightarrow{G}^{\int_0^t Ldt}.$$

Then, we are easily to generalize Taylor formula in $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ following.

Theorem 3.1(Taylor) *Let $\overrightarrow{G}^L \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ and there exist kth order derivative of L to t on a domain $\mathscr{D} \subset \mathbb{R}$, where $k \geq 1$. If A^+_{vu}, A^+_{uv} are linear for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, then*

$$\overrightarrow{G}^L = \overrightarrow{G}^{L(t_0)} + \frac{t-t_0}{1!}\overrightarrow{G}^{L'(t_0)} + \cdots + \frac{(t-t_0)^k}{k!}\overrightarrow{G}^{L^{(k)}(t_0)} + o\left((t-t_0)^{-k}\overrightarrow{G}\right), \qquad (3.1)$$

for $\forall t_0 \in \mathscr{D}$, where $o\left((t-t_0)^{-k}\overrightarrow{G}\right)$ denotes such an infinitesimal term $\widehat{L}$ of L that

$$\lim_{t\to t_0} \frac{\widehat{L}(v,u)}{(t-t_0)^k} = 0 \quad \text{for} \quad \forall (v,u) \in E\left(\overrightarrow{G}\right).$$

Particularly, if $L(v,u) = f(t)c_{vu}$, where c_{vu} is a constant, denoted by $f(t)\overrightarrow{G}^{L_C}$ with L_C : $(v,u) \to c_{vu}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and

$$f(t) = f(t_0) + \frac{(t-t_0)}{1!}f'(t_0) + \frac{(t-t_0)^2}{2!}f''(t_0) + \cdots + \frac{(t-t_0)^k}{k!}f^{(k)}(t_0) + o\left((t-t_0)^k\right),$$

then

$$f(t)\overrightarrow{G}^{L_C} = f(t)\cdot\overrightarrow{G}^{L_C}.$$

Proof Notice that $L(v,u)$ has kth order derivative to t on $\mathscr{D}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. By applying Taylor formula on t_0, we know that

$$L(v,u) = L(v,u)(t_0) + \frac{L'(v,u)(t_0)}{1!}(t-t_0) + \cdots + \frac{L^{(k)(v,u)(t_0)}}{k!} + o\left((t-t_0)^k\right)$$

if $t \to t_0$, where $o\left((t-t_0)^k\right)$ is an infinitesimal term $\widehat{L}(v,u)$ of $L(v,u)$ hold with

$$\lim_{t\to t_0} \frac{\widehat{L}(v,u)}{(t-t_0)^t} = 0$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. By operations (2.1) and (2.2),

$$\overrightarrow{G}^{L_1}+\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_1+L_2}\quad\text{and}\quad\lambda\overrightarrow{G}^{L}=\overrightarrow{G}^{\lambda\overrightarrow{L}}$$

because A^+_{vu}, A^+_{uv} are linear for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. We therefore get

$$\overrightarrow{G}^{L}=\overrightarrow{G}^{L(t_0)}+\frac{(t-t_0)}{1!}\overrightarrow{G}^{L'(t_0)}+\cdots+\frac{(t-t_0)^k}{k!}\overrightarrow{G}^{L^{(k)}(t_0)}+o\left((t-t_0)^{-k}\,\overrightarrow{G}\right)$$

for $t_0\in\mathscr{D}$, where $o\left((t-t_0)^{-k}\,\overrightarrow{G}\right)$ is an infinitesimal term $\widehat{L}$ of L, i.e.,

$$\lim_{t\to t_0}\frac{\widehat{L}(v,u)}{(t-t_0)^t}=0$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. Calculation also shows that

$$\begin{aligned}
f(t)\overrightarrow{G}^{L_C(v,u)} &= \overrightarrow{G}^{f(t)L_C(v,u)}=\overrightarrow{G}^{\left(f(t_0)+\frac{(t-t_0)}{1!}f'(t_0)\cdots+\frac{(t-t_0)^k}{k!}f^{(k)}(t_0)+o\left((t-t_0)^k\right)\right)c_{vu}}\\
&= f(t_0)c_{vu}\overrightarrow{G}+\frac{f'(t_0)(t-t_0)}{1!}c_{vu}\overrightarrow{G}+\frac{f"(t_0)(t-t_0)^2}{2!}c_{vu}\overrightarrow{G}\\
&\quad+\cdots+\frac{f^{(k)}(t_0)\,(t-t_0)^k}{k!}c_{vu}\overrightarrow{G}+o\left((t-t_0)^k\right)\overrightarrow{G}\\
&=\left(f(t_0)+\frac{(t-t_0)}{1!}f'(t_0)\cdots+\frac{(t-t_0)^k}{k!}f^{(k)}(t_0)+o\left((t-t_0)^k\right)\right)c_{vu}\overrightarrow{G}\\
&= f(t)c_{vu}\overrightarrow{G}=f(t)\cdot\overrightarrow{G}^{L_C(v,u)},
\end{aligned}$$

i.e.,

$$f(t)\overrightarrow{G}^{L_C}=f(t)\cdot\overrightarrow{G}^{L_C}.$$

This completes the proof. □

Taylor expansion formula for continuity flow $\overrightarrow{G}^{L}$ enables one to find interesting results on $\overrightarrow{G}^{L}$ such as those of the following.

Theorem 3.2 *Let $f(t)$ be a k differentiable function to t on a domain $\mathscr{D}\subset\mathbb{R}$ with $0\in\mathscr{D}$ and $f(0\overrightarrow{G})=f(0)\overrightarrow{G}$. If A^+_{vu}, A^+_{uv} are linear for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, then*

$$f(t)\overrightarrow{G}=f\left(t\overrightarrow{G}\right).\tag{3.2}$$

Proof Let $t_0=0$ in the Taylor formula. We know that

$$f(t)=f(0)+\frac{f'(0)}{1!}t+\frac{f''(0)}{2!}t^2+\cdots+\frac{f^{(k)}(0)}{k!}t^k+o\left(t^k\right).$$

Notice that

$$\begin{aligned} f(t)\overrightarrow{G} &= \left(f(0)+\frac{f'(0)}{1!}t+\frac{f''(0)}{2!}t^2+\cdots+\frac{f^{(k)}(0)}{k!}t^k+o\left(t^k\right)\right)\overrightarrow{G} \\ &= \overrightarrow{G}^{f(0)+\frac{f'(0)}{1!}t+\frac{f''(0)}{2!}t^2+\cdots+\frac{f^{(k)}(0)}{k!}t^k+o\left(t^k\right)} \\ &= f(0)\overrightarrow{G}+\frac{f'(0)t}{1!}\overrightarrow{G}+\cdots+\frac{f^{(k)}(0)t^k}{k!}\overrightarrow{G}+o\left(t^k\right)\overrightarrow{G} \end{aligned}$$

and by definition,

$$\begin{aligned} f\left(t\overrightarrow{G}\right) &= f\left(0\overrightarrow{G}\right)+\frac{f'(0)}{1!}\left(t\overrightarrow{G}\right)+\frac{f''(0)}{2!}\left(t\overrightarrow{G}\right)^2 \\ &\quad +\cdots+\frac{f^{(k)}(0)}{k!}\left(t\overrightarrow{G}\right)^k+o\left(\left(t\overrightarrow{G}\right)^k\right) \\ &= f\left(0\overrightarrow{G}\right)+\frac{f'(0)}{1!}t\overrightarrow{G}+\frac{f''(0)}{2!}t^2\overrightarrow{G}+\cdots+\frac{f^{(k)}(0)}{k!}t^k\overrightarrow{G}+o\left(t^k\right)\overrightarrow{G} \end{aligned}$$

because of $\left(t\overrightarrow{G}\right)^i=\overrightarrow{G}^{t^i}=t^i\overrightarrow{G}$ for any integer $1\leq i\leq k$. Notice that $f(0\overrightarrow{G})=f(0)\overrightarrow{G}$. We therefore get that

$$f(t)\overrightarrow{G}=f\left(t\overrightarrow{G}\right).$$

□

Theorem 3.2 enables one easily getting Taylor expansion formulas by $f\left(t\overrightarrow{G}\right)$. For example, let $f(t)=e^t$. Then

$$e^t\overrightarrow{G}=e^{t\overrightarrow{G}} \tag{3.3}$$

by Theorem 3.5. Notice that $(e^t)'=e^t$ and $e^0=1$. We know that

$$e^{t\overrightarrow{G}}=e^t\overrightarrow{G}=\overrightarrow{G}+\frac{t}{1!}\overrightarrow{G}+\frac{t^2}{2!}\overrightarrow{G}+\cdots+\frac{t^k}{k!}\overrightarrow{G}+\cdots \tag{3.4}$$

and

$$e^{t\overrightarrow{G}}\cdot e^{s\overrightarrow{G}}=\overrightarrow{G}^{e^t}\cdot\overrightarrow{G}^{e^s}=\overrightarrow{G}^{e^t\cdot e^s}=\overrightarrow{G}^{e^{t+s}}=e^{(t+s)\overrightarrow{G}}, \tag{3.5}$$

where t and s are variables, and similarly, for a real number α if $|t|<1$,

$$\left(\overrightarrow{G}+t\overrightarrow{G}\right)^\alpha=\overrightarrow{G}+\frac{\alpha t}{1!}\overrightarrow{G}+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)t^n}{n!}\overrightarrow{G}+\cdots \tag{3.6}$$

3.2 Limitation

Definition 3.3 *Let $\overrightarrow{G}^L,\overrightarrow{G}_1^{L_1}\in\left\langle\overrightarrow{G}_i,1\leq i\leq n\right\rangle^{\mathscr{V}}$ with L,L_1 dependent on a variable $t\in[a,b]\subset(-\infty,+\infty)$ and linear continuous end-operators A_{vu}^+ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. For $t_0\in[a,b]$ and any number $\varepsilon>0$, if there is always a number $\delta(\varepsilon)$ such that if $|t-t_0|\leq\delta(\varepsilon)$ then $\left\|\overrightarrow{G}_1^{L_1}-\overrightarrow{G}^L\right\|<\varepsilon$, then, $\overrightarrow{G}_1^{L_1}$ is said to be converged to $\overrightarrow{G}^L$ as $t\to t_0$, denoted by $\lim\limits_{t\to t_0}\overrightarrow{G}_1^{L_1}=\overrightarrow{G}^L$. Particularly, if $\overrightarrow{G}^L$ is a continuity flow with a constant $L(v)$ for $\forall v\in V\left(\overrightarrow{G}\right)$ and $t_0=+\infty$, $\overrightarrow{G}_1^{L_1}$ is said to be $\overrightarrow{G}$-synchronized.*

Applying Theorem 1.4, we know that there are positive constants $c_{vu} \in \mathbb{R}$ such that $\|A^+_{vu}\| \leq c^+_{vu}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

By definition, it is clear that

$$\begin{aligned}
&\left\|\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^L\right\| \\
&= \left\|\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)^{L_1}\right\| + \left\|\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)^{L_1 - L}\right\| + \left\|\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)^{-L}\right\| \\
&= \sum_{u \in N_{G_1 \setminus G}(v)} \left\|L_1^{A'^+_{vu}}(v,u)\right\| + \sum_{u \in N_{G_1 \bigcap G}(v)} \left\|\left(L_1^{A'^+_{vu}} - L_{vu}^{A^+_{vu}}\right)(v,u)\right\| + \sum_{u \in N_{G \setminus G_1}(v)} \left\|-L^{A^+_{vu}}(v,u)\right\| \\
&\leq \sum_{u \in N_{G_1 \setminus G}(v)} c^+_{vu}\|L_1(v,u)\| + \sum_{u \in N_{G_1 \bigcap G}(v)} c^+_{vu}\|\left(L_1 - L\right)(v,u)\| + \sum_{u \in N_{G \setminus G_1}(v)} c^+_{vu}\| - L(v,u)\|.
\end{aligned}$$

and $\|L(v,u)\| \geq 0$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ and $E\left(\overrightarrow{G}_1\right)$. Let

$$c^{\max}_{G_1 G} = \left\{\max_{(v,u) \in E(G_1)} c^+_{vu}, \max_{(v,u) \in E(G_1)} c^+_{vu}\right\}.$$

If $\left\|\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^L\right\| < \varepsilon$, we easily get that $\|L_1(v,u)\| < c^{\max}_{G_1G}\varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)$, $\|(L_1 - L)(v,u)\| < c^{\max}_{G_1G}\varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)$ and $\| - L(v,u)\| < c^{\max}_{G_1G}\varepsilon$ for $(v,u) \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)$.

Conversely, if $\|L_1(v,u)\| < \varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)$, $\|(L_1 - L)(v,u)\| < \varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)$ and $\| - L(v,u)\| < \varepsilon$ for $(v,u) \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)$, we easily know that

$$\begin{aligned}
\left\|\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^L\right\| \quad &= \sum_{u \in N_{G_1 \setminus G}(v)} \left\|L_1^{A'^+_{vu}}(v,u)\right\| + \sum_{u \in N_{G_1 \bigcap G}(v)} \left\|\left(L_1^{A'^+_{vu}} - L_{vu}^{A^+_{vu}}\right)(v,u)\right\| \\
&+ \sum_{u \in N_{G \setminus G_1}(v)} \left\|-L^{A^+_{vu}}(v,u)\right\| \\
&\leq \sum_{u \in N_{G_1 \setminus G}(v)} c^+_{vu}\|L_1(v,u)\| + \sum_{u \in N_{G_1 \bigcap G}(v)} c^+_{vu}\|\left(L_1 - L\right)(v,u)\| \\
&+ \sum_{u \in N_{G \setminus G_1}(v)} c^+_{vu}\| - L(v,u)\| \\
&< \left|\overrightarrow{G}_1 \setminus \overrightarrow{G}\right| c^{\max}_{G_1G}\varepsilon + \left|\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right| c^{\max}_{G_1G}\varepsilon + \left|\overrightarrow{G} \setminus \overrightarrow{G}_1\right| c^{\max}_{G_1G}\varepsilon = \left|\overrightarrow{G}_1 \bigcup \overrightarrow{G}\right| c^{\max}_{G_1G}\varepsilon.
\end{aligned}$$

Thus, we get an equivalent condition for $\lim_{t \to t_0} \overrightarrow{G}_1^{L_1} = \overrightarrow{G}^L$ following.

Theorem 3.4 $\lim_{t \to t_0} \overrightarrow{G}_1^{L_1} = \overrightarrow{G}^L$ *if and only if for any number* $\varepsilon > 0$ *there is always a number* $\delta(\varepsilon)$ *such that if* $|t - t_0| \leq \delta(\varepsilon)$ *then* $\|L_1(v,u)\| < \varepsilon$ *for* $(v,u) \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)$, $\|(L_1 - L)(v,u)\| < \varepsilon$ *for* $(v,u) \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)$ *and* $\| - L(v,u)\| < \varepsilon$ *for* $(v,u) \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)$,*i.e.,* $\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^L$ *is an infinitesimal or* $\lim_{t \to t_0} \left(\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^L\right) = \mathbf{O}$.

If $\lim\limits_{t\to t_0}\overrightarrow{G}^L$, $\lim\limits_{t\to t_0}\overrightarrow{G_1}^{L_1}$ and $\lim\limits_{t\to t_0}\overrightarrow{G_2}^{L_2}$ exist, the formulas following are clearly true by definition:

$$\begin{aligned}
&\lim_{t\to t_0}\left(\overrightarrow{G_1}^{L_1}+\overrightarrow{G_2}^{L_2}\right)=\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}+\lim_{t\to t_0}\overrightarrow{G_2}^{L_2},\\
&\lim_{t\to t_0}\left(\overrightarrow{G_1}^{L_1}\cdot\overrightarrow{G_2}^{L_2}\right)=\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}\cdot\lim_{t\to t_0}\overrightarrow{G_2}^{L_2},\\
&\lim_{t\to t_0}\left(\overrightarrow{G}^L\cdot\left(\overrightarrow{G_1}^{L_1}+\overrightarrow{G_2}^{L_2}\right)\right)=\lim_{t\to t_0}\overrightarrow{G}^L\cdot\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}+\lim_{t\to t_0}\overrightarrow{G}^L\cdot\lim_{t\to t_0}\overrightarrow{G_2}^{L_2},\\
&\lim_{t\to t_0}\left(\left(\overrightarrow{G_1}^{L_1}+\overrightarrow{G_2}^{L_2}\right)\cdot\overrightarrow{G}^L\right)=\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}\cdot\lim_{t\to t_0}\overrightarrow{G}^L+\lim_{t\to t_0}\overrightarrow{G_2}^{L_2}\cdot\lim_{t\to t_0}\overrightarrow{G}^L
\end{aligned}$$

and furthermore, if $\lim\limits_{t\to t_0}\overrightarrow{G_2}^{L_2}\neq\mathbf{O}$, then

$$\lim_{t\to t_0}\left(\frac{\overrightarrow{G_1}^{L_1}}{\overrightarrow{G_2}^{L_2}}\right)=\lim_{t\to t_0}\left(\overrightarrow{G_1}^{L_1}\cdot\overrightarrow{G_2}^{L_2^{-1}}\right)=\frac{\lim\limits_{t\to t_0}\overrightarrow{G_1}^{L_1}}{\lim\limits_{t\to t_0}\overrightarrow{G_2}^{L_2}}.$$

Theorem 3.5(L'Hospital's rule) *If* $\lim\limits_{t\to t_0}\overrightarrow{G_1}^{L_1}=\mathbf{O}$, $\lim\limits_{t\to t_0}\overrightarrow{G_2}^{L_2}=\mathbf{O}$ *and* L_1, L_2 *are differentiable respect to* t *with* $\lim\limits_{t\to t_0}L_1'(v,u)=0$ *for* $(v,u)\in E\left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)$, $\lim\limits_{t\to t_0}L_2'(v,u)\neq 0$ *for* $(v,u)\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)$ *and* $\lim\limits_{t\to t_0}L_2'(v,u)=0$ *for* $(v,u)\in E\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)$, *then,*

$$\lim_{t\to t_0}\left(\frac{\overrightarrow{G_1}^{L_1}}{\overrightarrow{G_2}^{L_2}}\right)=\frac{\lim\limits_{t\to t_0}\overrightarrow{G_1}^{L_1'}}{\lim\limits_{t\to t_0}\overrightarrow{G_2}^{L_2'}}.$$

Proof By definition, we know that

$$\begin{aligned}
\lim_{t\to t_0}\left(\frac{\overrightarrow{G_1}^{L_1}}{\overrightarrow{G_2}^{L_2}}\right) &= \lim_{t\to t_0}\left(\overrightarrow{G_1}^{L_1}\cdot\overrightarrow{G_2}^{L_2^{-1}}\right)\\
&= \lim_{t\to t_0}\left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)^{L_1}\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{L_1\cdot L_2^{-1}}\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)^{L_2}\\
&= \lim_{t\to t_0}\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{L_1\cdot L_2^{-1}}=\lim_{t\to t_0}\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{\frac{L_1}{L_2^{-1}}}\\
&= \left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{\lim\limits_{t\to t_0}\frac{L_1}{L_2^{-1}}}=\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{\frac{\lim\limits_{t\to t_0}{L'}_1}{\lim\limits_{t\to t_0}{L'}_2^{-1}}}\\
&= \left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)^{\lim\limits_{t\to t_0}L_1'}\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{\lim\limits_{t\to t_0}{L'}_1\cdot\lim\limits_{t\to t_0}{L'}_2^{-1}}\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)^{\lim\limits_{t\to t_0}L_2'}\\
&= \overrightarrow{G}_1^{\lim\limits_{t\to t_0}L_1'}\cdot\overrightarrow{G}_2^{\lim\limits_{t\to t_0}{L'}_2^{-1}}=\frac{\lim\limits_{t\to t_0}\overrightarrow{G}_1^{L_1'}}{\lim\limits_{t\to t_0}\overrightarrow{G}_2^{L_2'}}.
\end{aligned}$$

This completes the proof. □

Corollary 3.6 *If* $\lim\limits_{t\to t_0}\overrightarrow{G}^{L_1}=\mathbf{O}$, $\lim\limits_{t\to t_0}\overrightarrow{G}^{L_2}=\mathbf{O}$ *and* L_1, L_2 *are differentiable respect to* t *with* $\lim\limits_{t\to t_0}L_2'(v,u)\neq 0$ *for* $(v,u)\in E\left(\overrightarrow{G}\right)$, *then*

$$\lim_{t\to t_0}\left(\frac{\overrightarrow{G}^{L_1}}{\overrightarrow{G}^{L_2}}\right)=\frac{\lim\limits_{t\to t_0}\overrightarrow{G}^{L_1'}}{\lim\limits_{t\to t_0}\overrightarrow{G}^{L_2'}}.$$

Generally, by Taylor formula

$$\overrightarrow{G}^{L}=\overrightarrow{G}^{L(t_0)}+\frac{t-t_0}{1!}\overrightarrow{G}^{L'(t_0)}+\cdots+\frac{(t-t_0)^k}{k!}\overrightarrow{G}^{L^{(k)}(t_0)}+o\left((t-t_0)^{-k}\overrightarrow{G}\right),$$

if $L_1(t_0)=L_1'(t_0)=\cdots=L_1^{(k-1)}(t_0)=0$ and $L_2(t_0)=L_2'(t_0)=\cdots=L_2^{(k-1)}(t_0)=0$ but $L_2^{(k)}(t_0)\neq 0$, then

$$\begin{aligned}\overrightarrow{G}_1^{L_1} &= \frac{(t-t_0)^k}{k!}\overrightarrow{G}_1^{L_1^{(k)}(t_0)}+o\left((t-t_0)^{-k}\overrightarrow{G}_1\right),\\ \overrightarrow{G}_2^{L_2} &= \frac{(t-t_0)^k}{k!}\overrightarrow{G}_2^{L_2^{(k)}(t_0)}+o\left((t-t_0)^{-k}\overrightarrow{G}_2\right).\end{aligned}$$

We are easily know the following result.

Theorem 3.7 *If* $\lim\limits_{t\to t_0}\overrightarrow{G_1}^{L_1}=\mathbf{O}$, $\lim\limits_{t\to t_0}\overrightarrow{G_2}^{L_2}=\mathbf{O}$ *and* $L_1(t_0)=L_1'(t_0)=\cdots=L_1^{(k-1)}(t_0)=0$ *and* $L_2(t_0)=L_2'(t_0)=\cdots=L_2^{(k-1)}(t_0)=0$ *but* $L_2^{(k)}(t_0)\neq 0$, *then*

$$\lim_{t\to t_0}\frac{\overrightarrow{G}_1^{L_1}}{\overrightarrow{G}_2^{L_2}}=\frac{\lim\limits_{t\to t_0}\overrightarrow{G}_1^{L_1^{(k)}(t_0)}}{\lim\limits_{t\to t_0}\overrightarrow{G}_2^{L_2^{(k)}(t_0)}}.$$

Example 3.8 Let $\overrightarrow{G}_1=\overrightarrow{G}_2=\overrightarrow{C}_n$, $A^+_{v_iv_{i+1}}=1$, $A^+_{v_iv_{i-1}}=2$ and

$$f_i=\frac{f_1+\left(2^{i-1}-1\right)F(\overline{x})}{2^{i-1}}+\frac{n!}{(2n+1)e^t}$$

for integers $1\leq i\leq n$ in Fig.4.

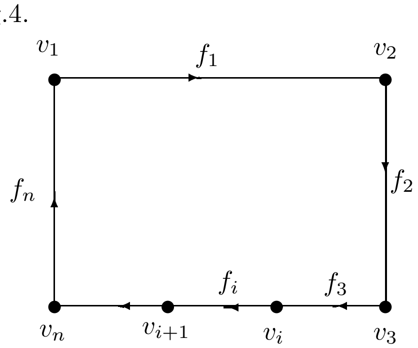

Fig.4

Calculation shows That

$$\begin{aligned} L(v_i) &= 2f_{i+1} - f_i = 2 \times \frac{f_1 + \left(2^i - 1\right) F(\overline{x})}{2^i} - \frac{f_1 + \left(2^{i-1} - 1\right) F(\overline{x})}{2^{i-1}} \\ &= F(\overline{x}) + \frac{n!}{(2n+1)e^t}. \end{aligned}$$

Calculation shows that $\lim\limits_{t\to\infty} L(v_i) = F(\overline{x})$, i.e., $\lim\limits_{t\to\infty} \overrightarrow{C}_n^L = \overrightarrow{C}_n^{\widehat{L}}$, where, $\widehat{L}(v_i) = F(\overline{x})$ for integers $1 \leq i \leq n$, i.e., $\overrightarrow{C}_n^L$ is $\overrightarrow{G}$-synchronized.

§4. Continuity Flow Equations

A continuity flow $\overrightarrow{G}^L$ is in fact an operator $L : \overrightarrow{G} \to \mathscr{B}$ determined by $L(v,u) \in \mathscr{B}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. Generally, let

$$[L]_{m\times n} = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ L_{m1} & L_{m2} & \cdots & L_{mn} \end{pmatrix}$$

with $L_{ij} : \overrightarrow{G} \to \mathscr{B}$ for $1 \leq i \leq m, 1 \leq j \leq n$, called operator matrix. Particularly, if for integers $1 \leq i \leq m, 1 \leq j \leq n$, $L_{ij} : \overrightarrow{G} \to \mathbb{R}$, we can also determine its rank as the usual, labeled the edge (v,u) by $\mathrm{Rank}[L]_{m\times n}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and get a labeled graph $\overrightarrow{G}^{\mathrm{Rank}[L]}$. Then we get a result following.

Theorem 4.1 *A linear continuity flow equations*

$$\left\{\begin{array}{l} x_1\overrightarrow{G}^{L_{11}} + x_2\overrightarrow{G}^{L_{12}} + \cdots + x_n\overrightarrow{G}^{L_{n1}} = \overrightarrow{G}^{L_1} \\ x_1\overrightarrow{G}^{L_{21}} + x_2\overrightarrow{G}^{L_{22}} + \cdots + x_n\overrightarrow{G}^{L_{2n}} = \overrightarrow{G}^{L_2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1\overrightarrow{G}^{L_{n1}} + x_2\overrightarrow{G}^{L_{n2}} + \cdots + x_n\overrightarrow{G}^{L_{nn}} = \overrightarrow{G}^{L_n} \end{array}\right. \tag{4.1}$$

is solvable if and only if

$$\overrightarrow{G}^{\mathrm{Rank}[L]} = \overrightarrow{G}^{\mathrm{Rank}\left[\overline{L}\right]}, \tag{4.2}$$

where

$$[L] = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix} \quad \textit{and} \quad \left[\overline{L}\right] = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} & L_1 \\ L_{21} & L_{22} & \cdots & L_{2n} & L_2 \\ \cdots & \cdots & \cdots & \cdots & \\ L_{n1} & L_{n2} & \cdots & L_{nn} & L_n \end{pmatrix}.$$

Proof Clearly, if (4.1) is solvable, then for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, the linear equations

$$\left\{\begin{array}{l} x_1L_{11}(v,u)+x_2L_{12}(v,u)+\cdots+x_nL_{n1}(v,u0=L_1(v,u)\\ x_1L_{21}(v,u)+x_2L_{22}(v,u)+\cdots+x_nL_{21}(v,u0=L_2(v,u)\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_1L_{n1}(v,u)+x_2L_{n2}(v,u)+\cdots+x_nL_{nn}(v,u0=L_n(v,u)\end{array}\right.$$

is solvable. By linear algebra, there must be

$$\text{Rank}\left(\begin{array}{cccc} L_{11}(v,u) & L_{12}(v,u) & \cdots & L_{1n}(v,u)\\ L_{21}(v,u) & L_{22}(v,u) & \cdots & L_{2n}(v,u)\\ \cdots & \cdots & \cdots & \cdots\\ L_{n1}(v,u) & L_{n2}(v,u) & \cdots & L_{nn}(v,u)\end{array}\right)=$$

$$\text{Rank}\left(\begin{array}{ccccc} L_{11}(v,u) & L_{12}(v,u) & \cdots & L_{1n}(v,u) & L_1(v,u)\\ L_{21}(v,u) & L_{22}(v,u) & \cdots & L_{2n}(v,u) & L_2(v,u)\\ \cdots & \cdots & \cdots & \cdots & \\ L_{n1}(v,u) & L_{n2}(v,u) & \cdots & L_{nn}(v,u) & L_n(v,u)\end{array}\right),$$

which implies that

$$\overrightarrow{G}^{\text{Rank}[L]}=\overrightarrow{G}^{\text{Rank}\left[\overline{L}\right]}.$$

Conversely, if the (4.2) is hold, then for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, the linear equations

$$\left\{\begin{array}{l} x_1L_{11}(v,u)+x_2L_{12}(v,u)+\cdots+x_nL_{n1}(v,u0=L_1(v,u)\\ x_1L_{21}(v,u)+x_2L_{22}(v,u)+\cdots+x_nL_{21}(v,u0=L_2(v,u)\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_1L_{n1}(v,u)+x_2L_{n2}(v,u)+\cdots+x_nL_{nn}(v,u0=L_n(v,u)\end{array}\right.$$

is solvable, i.e., the equations (4.1) is solvable. □

Theorem 4.2 *A continuity flow equation*

$$\lambda^s\overrightarrow{G}^{L_s}+\lambda^{s-1}\overrightarrow{G}^{L_{s-1}}+\cdots+\overrightarrow{G}^{L_0}=\mathbf{O} \tag{4.3}$$

always has solutions $\overrightarrow{G}^{L_\lambda}$ *with* $L_\lambda:(v,u)\in E\left(\overrightarrow{G}\right)\to\{\lambda_1^{vu},\lambda_2^{vu},\cdots,\lambda_s^{vu}\}$, *where* $\lambda_i^{vu}, 1\le i\le s$ *are roots of the equation*

$$\alpha_s^{vu}\lambda^s+\alpha_{s-1}^{vu}\lambda^{s-1}+\cdots+\alpha_0^{vu}=0 \tag{4.4}$$

with $L_i:(v,u)\to\alpha_i^{v,u}$, $\alpha_s^{vu}\ne 0$ *a constant for* $(v,u)\in E\left(\overrightarrow{G}\right)$ *and* $1\le i\le s$.

For $(v,u)\in E\left(\overrightarrow{G}\right)$, *if* n^{vu} *is the maximum number* i *with* $L_i(v,u)\ne 0$, *then there are*

$\prod\limits_{(v,u)\in E\left(\overrightarrow{G}\right)} n^{vu}$ *solutions* $\overrightarrow{G}^{L_\lambda}$*, and particularly, if* $L_s(v,u)\neq 0$ *for* $\forall(v,u)\in E\left(\overrightarrow{G}\right)$*, there are* $s^{\left|E\left(\overrightarrow{G}\right)\right|}$ *solutions* $\overrightarrow{G}^{L_\lambda}$ *of equation* (4.3).

Proof By the fundamental theorem of algebra, we know there are s roots $\lambda_1^{vu},\lambda_2^{vu},\cdots,\lambda_s^{vu}$ for the equation (4.3). Whence, $L_\lambda\overrightarrow{G}$ is a solution of equation (4.2) because of

$$\begin{aligned}&\left(\lambda\overrightarrow{G}\right)^s\cdot\overrightarrow{G}^{L_s}+\left(\lambda\overrightarrow{G}\right)^{s-1}\cdot\overrightarrow{G}^{L_{s-1}}+\cdots+\left(\lambda\overrightarrow{G}\right)^0\cdot\overrightarrow{G}^{L_0}\\&=\overrightarrow{G}^{\lambda^sL_s}+\overrightarrow{G}^{\lambda^{s-1}L_{s-1}}+\cdots+\overrightarrow{G}^{\lambda^0L_0}=\overrightarrow{G}^{\lambda^sL_s+\lambda^{s-1}L_{s-1}+\cdots+L_0}\end{aligned}$$

and

$$\lambda^sL_s+\lambda^{s-1}L_{s-1}+\cdots+L_0:\ (v,u)\to\alpha_s^{vu}\lambda^s+\alpha_{s-1}^{vu}\lambda^{s-1}+\cdots+\alpha_0^{vu}=0,$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, i.e.,

$$\left(\lambda\overrightarrow{G}\right)^s\cdot\overrightarrow{G}^{L_s}+\left(\lambda\overrightarrow{G}\right)^{s-1}\cdot\overrightarrow{G}^{L_{s-1}}+\cdots+\left(\lambda\overrightarrow{G}\right)^0\cdot\overrightarrow{G}^{L_0}=0\overrightarrow{G}=\mathbf{O}.$$

Count the number of different L_λ for $(v,u)\in E\left(\overrightarrow{G}\right)$. It is nothing else but just n^{vu}. Therefore, the number of solutions of equation (4.3) is $\prod\limits_{(v,u)\in E\left(\overrightarrow{G}\right)} n^{vu}$. □

Theorem 4.3 *A continuity flow equation*

$$\frac{d\overrightarrow{G}^L}{dt}=\overrightarrow{G}^{L_\alpha}\cdot\overrightarrow{G}^L \tag{4.5}$$

with initial values $\left.\overrightarrow{G}^L\right|_{t=0}=\overrightarrow{G}^{L_\beta}$ *always has a solution*

$$\overrightarrow{G}^L=\overrightarrow{G}^{L_\beta}\cdot\left(e^{tL_\alpha}\overrightarrow{G}\right),$$

where $L_\alpha:(v,u)\to\alpha_{vu},\ L_\beta:(v,u)\to\beta_{vu}$ *are constants for* $\forall(v,u)\in E\left(\overrightarrow{G}\right)$.

Proof A calculation shows that

$$\overrightarrow{G}^{\frac{dL}{dt}}=\frac{d\overrightarrow{G}^L}{dt}=\overrightarrow{G}^{L_\alpha}\cdot\overrightarrow{G}^L=\overrightarrow{G}^{L_\alpha\cdot L},$$

which implies that

$$\frac{dL}{dt}=\alpha_{vu}L \tag{4.6}$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$.

Solving equation (4.6) enables one knowing that $L(v,u)=\beta_{vu}e^{t\alpha_{vu}}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$.

Whence, the solution of (4.5) is

$$\overrightarrow{G}^{L} = \overrightarrow{G}^{L_\beta e^{tL_\alpha}} = \overrightarrow{G}^{L_\beta} \cdot \left(e^{tL_\alpha}\overrightarrow{G}\right)$$

and conversely, by Theorem 3.2,

$$\begin{aligned}\frac{d\overrightarrow{G}^{L_\beta e^{tL_\alpha}}}{dt} &= \overrightarrow{G}^{\frac{d\left(L_\beta e^{tL_\alpha}\right)}{dt}} = \overrightarrow{G}^{L_\alpha L_\beta e^{tL_\alpha}} \\ &= \overrightarrow{G}^{L_\alpha} \cdot \overrightarrow{G}^{L_\beta e^{tL_\alpha}},\end{aligned}$$

i.e.,

$$\frac{d\overrightarrow{G}^{L}}{dt} = \overrightarrow{G}^{L_\alpha} \cdot \overrightarrow{G}^{L}$$

if $\overrightarrow{G}^{L} = \overrightarrow{G}^{L_\beta} \cdot \left(e^{tL_\alpha}\overrightarrow{G}\right)$. This completes the proof. □

Theorem 4.3 can be generalized to the case of $L : (v,u) \to \mathbb{R}^n, n \geq 2$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

Theorem 4.4 *A complex flow equation*

$$\frac{d\overrightarrow{G}^{L}}{dt} = \overrightarrow{G}^{L_\alpha} \cdot \overrightarrow{G}^{L} \tag{4.7}$$

with initial values $\left.\overrightarrow{G}^{L}\right|_{t=0} = \overrightarrow{G}^{L_\beta}$ *always has a solution*

$$\overrightarrow{G}^{L} = \overrightarrow{G}^{L_\beta} \cdot \left(e^{tL_\alpha}\overrightarrow{G}\right),$$

where $L_\alpha : (v,u) \to \left(\alpha_{vu}^1, \alpha_{vu}^2, \cdots, \alpha_{vu}^n\right)$, $L_\beta : (v,u) \to \left(\beta_{vu}^1, \beta_{vu}^2, \cdots, \beta_{vu}^n\right)$ *with constants* α_{vu}^i, β_{vu}^i, $1 \leq i \leq n$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

Theorem 4.5 *A complex flow equation*

$$\overrightarrow{G}^{L_{\alpha_n}} \cdot \frac{d^n\overrightarrow{G}^{L}}{dt^n} + \overrightarrow{G}^{L_{\alpha_{n-1}}} \cdot \frac{d^{n-1}\overrightarrow{G}^{L}}{dt^{n-1}} + \cdots + \overrightarrow{G}^{L_{\alpha_0}} \cdot \overrightarrow{G}^{L} = \mathbf{O} \tag{4.8}$$

with $L_{\alpha_i} : (v,u) \to \alpha_i^{vu}$ *constants for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ *and integers* $0 \leq i \leq n$ *always has a general solution* $\overrightarrow{G}^{L_\lambda}$ *with*

$$L_\lambda : (v,u) \to \left\{ 0,\ \sum_{i=1}^{s} h_i(t) e^{\lambda_i^{vu} t} \right\}$$

for $(v,u) \in E\left(\overrightarrow{G}\right)$, *where* $h_{m_i}(t)$ *is a polynomial of degree* $\leq m_i - 1$ *on* t, $m_1 + m_2 + \cdots + m_s = n$ *and* $\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_s^{vu}$ *are the distinct roots of characteristic equation*

$$\alpha_n^{vu}\lambda^n + \alpha_{n-1}^{vu}\lambda^{n-1} + \cdots + \alpha_0^{vu} = 0$$

with $\alpha_n^{vu} \neq 0$ *for* $(v,u) \in E\left(\overrightarrow{G}\right)$.

Proof Clearly, the equation (4.8) on an edge $(v,u) \in E\left(\overrightarrow{G}\right)$ is

$$\alpha_n^{vu}\frac{d^nL(v,u)}{dt^n} + \alpha_{n-1}^{vu}\frac{d^{n-1}L(v,u)}{dt^{n-1}} + \cdots + \alpha_0 = 0. \tag{4.9}$$

As usual, assuming the solution of (4.6) has the form $\overrightarrow{G}^L = e^{\lambda t}\overrightarrow{G}$. Calculation shows that

$$\begin{aligned}\frac{d\overrightarrow{G}^L}{dt} &= \lambda e^{\lambda t}\overrightarrow{G} = \lambda\overrightarrow{G},\\ \frac{d^2\overrightarrow{G}^L}{dt^2} &= \lambda^2 e^{\lambda t}\overrightarrow{G} = \lambda^2\overrightarrow{G},\\ \cdots\cdots &\cdots \cdots\cdots\cdots\cdots\cdots,\\ \frac{d^n\overrightarrow{G}^L}{dt^n} &= \lambda^n e^{\lambda t}\overrightarrow{G} = \lambda^n\overrightarrow{G}.\end{aligned}$$

Substituting these calculation results into (4.8), we get that

$$\left(\lambda^n\overrightarrow{G}^{L_{\alpha_n}} + \lambda^{n-1}\overrightarrow{G}^{L_{\alpha_{n-1}}} + \cdots + \overrightarrow{G}^{L_{\alpha_0}}\right)\cdot\overrightarrow{G}^L = \mathbf{O},$$

i.e.,

$$\overrightarrow{G}^{\left(\lambda^n\cdot L_{\alpha_n}+\lambda^{n-1}\cdot L_{\alpha_{n-1}}+\cdots+L_{\alpha_0}\right)\cdot L} = \mathbf{O},$$

which implies that for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$,

$$\lambda^n\alpha_n^{vu} + \lambda^{n-1}\alpha_{n-1}^{vu} + \cdots + \alpha_0 = 0 \tag{4.10}$$

or

$$L(v,u) = 0.$$

Let $\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_s^{vu}$ be the distinct roots with respective multiplicities $m_1^{vu}, m_2^{vu}, \cdots, m_s^{vu}$ of equation (4.8). We know the general solution of (4.9) is

$$L(v,u) = \sum_{i=1}^{s} h_i(t)e^{\lambda_i^{vu}t}$$

with $h_{m_i}(t)$ a polynomial of degree$\leq m_i - 1$ on t by the theory of ordinary differential equations. Therefore, the general solution of (4.8) is $\overrightarrow{G}^{L_\lambda}$ with

$$L_\lambda : (v,u) \to \left\{\ 0,\ \sum_{i=1}^{s} h_i(t)e^{\lambda_i^{vu}t}\ \right\}$$

for $(v,u) \in E\left(\overrightarrow{G}\right)$. □

§5. Complex Flow with Continuity Flows

The difference of a complex flow $\overrightarrow{G}^L$ with that of a continuity flow $\overrightarrow{G}^L$ is the labeling L on a vertex is $L(v) = \dot{x}_v$ or x_v. Notice that

$$\frac{d}{dt}\left(\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u)\right) = \sum_{u\in N_G(v)} \frac{d}{dt} L^{A^+_{vu}}(v,u)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$. There must be relations between complex flows $\overrightarrow{G}^L$ and continuity flows $\overrightarrow{G}^L$. We get a general result following.

Theorem 5.1 *If end-operators* A^+_{vu}, A^+_{uv} *are linear with* $\left[\int_0^t, A^+_{vu}\right] = \left[\int_0^t, A^+_{uv}\right] = \mathbf{0}$ *and* $\left[\frac{d}{dt}, A^+_{vu}\right] = \left[\frac{d}{dt}, A^+_{uv}\right] = \mathbf{0}$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, *and particularly,* $A^+_{vu} = \mathbf{1}_{\mathscr{V}}$, *then* $\overrightarrow{G}^L \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ *is a continuity flow with a constant* $L(v)$ *for* $\forall v \in V\left(\overrightarrow{G}\right)$ *if and only if* $\int_0^t \overrightarrow{G}^L dt$ *is such a continuity flow with a constant one each vertex* v, $v \in V\left(\overrightarrow{G}\right)$.

Proof Notice that if $A^+_{vu} = \mathbf{1}_{\mathscr{V}}$, there always is $\left[\int_0^t, A^+_{vu}\right] = \mathbf{0}$ and $\left[\frac{d}{dt}, A^+_{vu}\right] = \mathbf{0}$, and by definition, we know that

$$\left[\int_0^t, A^+_{vu}\right] = \mathbf{0} \quad \Leftrightarrow \quad \int_0^t \circ A^+_{vu} = A^+_{vu} \circ \int_0^t,$$
$$\left[\frac{d}{dt}, A^+_{vu}\right] = \mathbf{0} \quad \Leftrightarrow \quad \frac{d}{dt} \circ A^+_{vu} = A^+_{vu} \circ \frac{d}{dt}.$$

If $\overrightarrow{G}^L$ is a continuity flow with a constant $L(v)$ for $\forall v \in V\left(\overrightarrow{G}\right)$, i.e.,

$$\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u) = \mathbf{v} \quad \text{for} \quad \forall v \in V\left(\overrightarrow{G}\right),$$

we are easily know that

$$\begin{aligned}\int_0^t \left(\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u)\right) dt &= \sum_{u\in N_G(v)} \left(\int_0^t \circ A^+_{vu}\right) L(v,u)dt = \sum_{u\in N_G(v)} \left(A^+_{vu} \circ \int_0^t\right) L(v,u)dt \\ &= \sum_{u\in N_G(v)} A^+_{vu}\left(\int_0^t L(v,u)dt\right) = \int_0^t \mathbf{v}dt\end{aligned}$$

for $\forall v \in V\left(\overrightarrow{G}\right)$ with a constant vector $\int_0^t \mathbf{v}dt$, i.e., $\int_0^t \overrightarrow{G}^L dt$ is a continuity flow with a constant flow on each vertex v, $v \in V\left(\overrightarrow{G}\right)$.

Conversely, if $\int_0^t \overrightarrow{G}^L dt$ is a continuity flow with a constant flow on each vertex v, $v \in$

$V\left(\overrightarrow{G}\right)$, i.e.,

$$\sum_{u\in N_G(v)} A^+_{vu}\circ\int_0^t L(v,u)dt = \mathbf{v} \ \text{ for } \ \forall v\in V\left(\overrightarrow{G}\right),$$

then

$$\overrightarrow{G}^L = \frac{d\left(\int_0^t \overrightarrow{G}^L dt\right)}{dt}$$

is such a continuity flow with a constant flow on vertices in $\overrightarrow{G}$ because of

$$\begin{aligned}\frac{d\left(\sum\limits_{u\in N_G(v)} L^{A^+_{vu}}(v,u)\right)}{dt} &= \sum_{u\in N_G(v)} \frac{d}{dt}\circ A^+_{vu}\circ\int_0^t L(v,u)dt \\ &= \sum_{u\in N_G(v)} A^+_{vu}\circ\frac{d}{dt}\circ\int_0^t L(v,u)dt = \sum_{u\in N_G(v)} L(v,u)^{A^+_{vu}} = \frac{d\mathbf{v}}{dt}\end{aligned}$$

with a constant flow $\frac{d\mathbf{v}}{dt}$ on vertex v, $v\in V\left(\overrightarrow{G}\right)$. This completes the proof. □

If all end-operators A^+_{vu} and A^+_{uv} are constant for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, the conditions $\left[\int_0^t, A^+_{vu}\right] = \left[\int_0^t, A^+_{uv}\right] = \mathbf{0}$ and $\left[\frac{d}{dt}, A^+_{vu}\right] = \left[\frac{d}{dt}, A^+_{uv}\right] = \mathbf{0}$ are clearly true. We immediately get a conclusion by Theorem 5.1 following.

Corollary 5.2 *For $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, if end-operators A^+_{vu} and A^+_{uv} are constant c_{vu}, c_{uv} for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, then $\overrightarrow{G}^L\in\left\langle\overrightarrow{G}_i, 1\le i\le n\right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ is a continuity flow with a constant $L(v)$ for $\forall v\in V\left(\overrightarrow{G}\right)$ if and only if $\int_0^t \overrightarrow{G}^L dt$ is such a continuity flow with a constant flow on each vertex v, $v\in V\left(\overrightarrow{G}\right)$.*

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