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Intrinsic Geometry of the Special Equations in Galilean 3−Space G_3

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Abstract: In this study, we investigate a general intrinsic geometry in 3-dimensional Galilean space G_3. Then, we obtain some special equations by using intrinsic derivatives of orthonormal triad in G_3.

Key Words: NLS Equation, Galilean Space.

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§1. Introduction

A Galilean space is a three dimensional complex projective space, where $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), real line $f \subset w$ (the absolute line) and two complex conjugate points I_1, $I_2 \in f$ (the absolute points). We shall take as a real model of the space G_3, a real projective space P_3 with the absolute $\{w, f\}$ consisting of a real plane $w \subset G_3$ and a real line $f \subset w$ on which an elliptic involution ε has been defined. The Galilean scalar product between two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ is defined [3]

$$(a.b)_G = \begin{cases} a_1b_1, & if\ a_1 \neq 0\ or\ b_1 \neq 0, \\ a_2b_2 + a_3b_3, & if\ \ a_1 = b_1 = 0. \end{cases}$$

and the Galilean vector product is defined

$$(a \wedge b)_G = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, & if\ a_1 \neq 0\ or\ b_1 \neq 0, \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, & if\ a_1 = b_1 = 0. \end{cases}$$

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Let $\alpha : I \rightarrow G_3,\ I \subset R$ be an unit speed curve in Galilean space G_3 parametrized by the invariant parameter $s \in I$ and given in the coordinate form
$\alpha(s) = (s, y(s), z(s))$. Then the curvature and the torsion of the curve α are given by

$$\kappa(s) = \|\alpha''(s)\|, \quad \tau(s) = \frac{1}{\kappa^2(s)} Det\left(\alpha'(s), \alpha''(s), \alpha'''(s)\right)$$

respectively. The Frenet frame $\{t, n, b\}$ of the curve α is given by

$$\begin{aligned} t(s) &= \alpha'(s) = (1, y'(s), z'(s)), \\ n(s) &= \frac{\alpha''(s)}{\|\alpha''(s)\|} = \frac{1}{\kappa(s)}(1, y''(s), z''(s)), \\ b(s) &= (t(s) \wedge n(s))_G = \frac{1}{\kappa(s)}(1, -z''(s), y''(s)), \end{aligned}$$

where $t(s)$, $n(s)$ and $b(s)$ are called the tangent vector, principal normal vector and binormal vector, respectively. The Frenet formulas for $\alpha(s)$ given by [3] are

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ 0 & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}. \tag{1.1}$$

The binormal motion of curves in the Galilean 3-space is equivalent to the nonlinear Schrödinger equation (NLS$^-$) of repulsive type

$$iq_b + q_{ss} - \frac{1}{2}\left|\langle q, q\rangle\right|^2 \bar{q} = 0 \tag{1.2}$$

where

$$q = \kappa \exp\left(\int_0^s \sigma ds\right), \quad \sigma = \kappa \exp\left(\int_0^s r ds\right). \tag{1.3}$$

§2. Basic Properties of Intrinsic Geometry

Intrinsic geometry of the nonlinear Schrodinger equation was investigated in E^3 by Rogers and Schief. According to anholonomic coordinates, characterization of three dimensional vector field was introduced in E^3 by Vranceau [5], and then analyse Marris and Passman [3].

Let ϕ be a 3-dimensional vector field according to anholonomic coordinates in G_3. The $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ is the tangent, principal normal and binormal directions to the vector lines of ϕ. Intrinsic derivatives of this orthonormal triad are given by following

$$\frac{\delta}{\delta s}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{2.1}$$

$$\frac{\delta}{\delta n}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & \theta_{ns} & (\Omega_b+\tau) \\ -\theta_{ns} & 0 & -div\boldsymbol{b} \\ -(\Omega_b+\tau) & div\boldsymbol{b} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{2.2}$$

$$\frac{\delta}{\delta b}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & -(\Omega_n+\tau) & \theta_{bs} \\ (\Omega_n+\tau) & 0 & div\boldsymbol{n} \\ -\theta_{bs} & -div\boldsymbol{n} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}, \tag{2.3}$$

where $\frac{\delta}{\delta s}$, $\frac{\delta}{\delta n}$ and $\frac{\delta}{\delta b}$ are directional derivatives in the tangential, principal normal and binormal directions in G_3. Thus, the equation (2.1) show the usual Serret-Frenet relations, also (2.2) and (2.3) give the directional derivatives of the orthonormal triad $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ in the n- and b-directions, respectively. Accordingly,

$$grad = \boldsymbol{t}\frac{\delta}{\delta s} + \boldsymbol{n}\frac{\delta}{\delta n} + \boldsymbol{b}\frac{\delta}{\delta b}, \tag{2.4}$$

where θ_{bs} and θ_{ns} are the quantities originally introduced by Bjorgum in 1951 [2] via

$$\theta_{ns} = \boldsymbol{n}\cdot\frac{\delta \boldsymbol{t}}{\delta n}, \qquad \theta_{bs} = \boldsymbol{b}\cdot\frac{\delta \boldsymbol{t}}{\delta b}. \tag{2.5}$$

From the usual Serret Frenet relations in G_3, we obtain the following equations

$$div\boldsymbol{t} = (\boldsymbol{t}\frac{\delta}{\delta s} + \boldsymbol{n}\frac{\delta}{\delta n} + \boldsymbol{b}\frac{\delta}{\delta b})\boldsymbol{t} = \boldsymbol{t}(\kappa\boldsymbol{n}) + \boldsymbol{n}\frac{\delta \boldsymbol{t}}{\delta n} + \boldsymbol{b}\frac{\delta \boldsymbol{t}}{\delta b} = \theta_{ns} + \theta_{bs}, \tag{2.6}$$

$$div\boldsymbol{n} = (\boldsymbol{t}\frac{\delta}{\delta s} + \boldsymbol{n}\frac{\delta}{\delta n} + \boldsymbol{b}\frac{\delta}{\delta b})\boldsymbol{n} = \boldsymbol{t}(\tau\boldsymbol{b}) + \boldsymbol{n}\frac{\delta \boldsymbol{n}}{\delta n} + \boldsymbol{b}\frac{\delta \boldsymbol{n}}{\delta b} = \boldsymbol{b}\frac{\delta \boldsymbol{n}}{\delta b}, \tag{2.7}$$

$$div\boldsymbol{b} = (\boldsymbol{t}\frac{\delta}{\delta s} + \boldsymbol{n}\frac{\delta}{\delta n} + \boldsymbol{b}\frac{\delta}{\delta b})\boldsymbol{b} = \boldsymbol{t}(-\tau\boldsymbol{n}) + \boldsymbol{n}\frac{\delta \boldsymbol{b}}{\delta n} + \boldsymbol{b}\frac{\delta \boldsymbol{b}}{\delta b} = \boldsymbol{n}\frac{\delta \boldsymbol{b}}{\delta n}. \tag{2.8}$$

Moreover, we get

$$\begin{aligned} curl\boldsymbol{t} &= \left(\boldsymbol{t}\times\frac{\delta}{\delta s} + \boldsymbol{n}\times\frac{\delta}{\delta n} + \boldsymbol{b}\times\frac{\delta}{\delta b}\right)\boldsymbol{t} \\ &= \boldsymbol{t}\times(\kappa\boldsymbol{n}) + \boldsymbol{n}\times\frac{\delta \boldsymbol{t}}{\delta n} + \boldsymbol{b}\times\frac{\delta \boldsymbol{t}}{\delta b} \\ &= \left[\frac{\delta \boldsymbol{t}}{\delta n}\boldsymbol{b} - \frac{\delta \boldsymbol{t}}{\delta b}\boldsymbol{n}\right](1,0,0) + \kappa\boldsymbol{b} \\ &\Rightarrow curl\boldsymbol{t} = \Omega_s\,(1,0,0) + \kappa\boldsymbol{b}, \end{aligned} \tag{2.9}$$

where

$$\Omega_s = \boldsymbol{t}\cdot curl\boldsymbol{t} = \boldsymbol{b}\cdot\frac{\delta \boldsymbol{t}}{\delta n} - \boldsymbol{n}\cdot\frac{\delta \boldsymbol{t}}{\delta b} \tag{2.10}$$

is defined the abnormality of the $\boldsymbol{t}$-field. Firstly, the relation (2.9) was obtained in E^3 by

Masotti. Also, we find

$$\begin{aligned} curl\boldsymbol{n} &= \left(\boldsymbol{t}\times\frac{\delta}{\delta s}+\boldsymbol{n}\times\frac{\delta}{\delta n}+\boldsymbol{b}\times\frac{\delta}{\delta b}\right)\boldsymbol{n} \\ &= \boldsymbol{t}\times(\tau\boldsymbol{b})+\boldsymbol{n}\times\frac{\delta\boldsymbol{n}}{\delta n}+\boldsymbol{b}\times\frac{\delta\boldsymbol{n}}{\delta b} \\ &= \left[\boldsymbol{t}\cdot\frac{\delta\boldsymbol{n}}{\delta b}-\tau\right]\boldsymbol{n}+\left(\boldsymbol{b}\frac{\delta\boldsymbol{n}}{\delta n}\right)(1,0,0)-\left(\boldsymbol{t}\frac{\delta\boldsymbol{n}}{\delta n}\right)\boldsymbol{b} \end{aligned}$$

$$\Rightarrow curl\boldsymbol{n}=-\left(div\boldsymbol{b}\right)(1,0,0)+\Omega_n\boldsymbol{n}+\theta_{ns}\boldsymbol{b}, \tag{2.11}$$

where

$$\Omega_n=\boldsymbol{n}\cdot curl\boldsymbol{n}=\boldsymbol{t}\cdot\frac{\delta\boldsymbol{n}}{\delta b}-\tau \tag{2.12}$$

is defined the abnormality of the $\boldsymbol{n}$-field and

$$\begin{aligned} curl\boldsymbol{b} &= \left(\boldsymbol{t}\times\frac{\delta}{\delta s}+\boldsymbol{n}\times\frac{\delta}{\delta n}+\boldsymbol{b}\times\frac{\delta}{\delta b}\right)\boldsymbol{b} \\ &= \boldsymbol{t}\times(-\tau\boldsymbol{n})+\boldsymbol{n}\times\left[\left(\boldsymbol{t}\frac{\delta\boldsymbol{b}}{\delta n}\right)\boldsymbol{t}\right]+\boldsymbol{b}\times\left[\left(\boldsymbol{t}\frac{\delta\boldsymbol{b}}{\delta b}\right)\boldsymbol{t}+\left(\boldsymbol{n}\frac{\delta\boldsymbol{b}}{\delta b}\right)\boldsymbol{n}\right] \\ &= -\left[\tau+\boldsymbol{t}\cdot\frac{\delta\boldsymbol{b}}{\delta n}\right]\boldsymbol{b}+\left(\boldsymbol{t}\frac{\delta\boldsymbol{b}}{\delta b}\right)\boldsymbol{n}+\left(\boldsymbol{b}\frac{\delta\boldsymbol{n}}{\delta b}\right)(1,0,0), \end{aligned}$$

$$\Rightarrow curl\boldsymbol{b}=\Omega_b\boldsymbol{b}-\theta_{bs}\boldsymbol{n}+\left(div\boldsymbol{n}\right)(1,0,0), \tag{2.13}$$

where

$$\Omega_b=\boldsymbol{b}\cdot curl\boldsymbol{b}=-\left(\tau+\boldsymbol{t}\cdot\frac{\delta\boldsymbol{b}}{\delta n}\right) \tag{2.14}$$

is defined the abnormality of the $\boldsymbol{b}$-field. By using the identity $curlgrad\varphi=0$, we have

$$\begin{aligned} &\left(\frac{\delta^2\varphi}{\delta n\delta b}-\frac{\delta^2\varphi}{\delta b\delta n}\right)\boldsymbol{t}+\left(\frac{\delta^2\varphi}{\delta b\delta s}-\frac{\delta^2\varphi}{\delta s\delta b}\right)\boldsymbol{n}+\left(\frac{\delta^2\varphi}{\delta s\delta n}-\frac{\delta^2\varphi}{\delta n\delta s}\right)\boldsymbol{b} \\ &+\frac{\delta\varphi}{\delta s}curl\boldsymbol{t}+\frac{\delta\varphi}{\delta n}curl\boldsymbol{n}+\frac{\delta\varphi}{\delta b}curl\boldsymbol{b}=0. \end{aligned} \tag{2.15}$$

Substituting (2.9), (2.11) and (2.13) in (2.15), we find

$$\begin{aligned} \frac{\delta^2\phi}{\delta n\delta b}-\frac{\delta^2\phi}{\delta n\delta b} &= -\frac{\delta\phi}{\delta s}\Omega_s+\frac{\delta\phi}{\delta n}(div\boldsymbol{b})-\frac{\delta\phi}{\delta b}(div\boldsymbol{n}) \\ \frac{\delta^2\phi}{\delta b\delta s}-\frac{\delta^2\phi}{\delta s\delta b} &= -\frac{\delta\phi}{\delta n}\Omega_n+\frac{\delta\phi}{\delta b}\theta_{bs} \\ \frac{\delta^2\phi}{\delta s\delta n}-\frac{\delta^2\phi}{\delta n\delta s} &= -\frac{\delta\phi}{\delta s}\kappa-\frac{\delta\phi}{\delta n}\theta_{ns}-\frac{\delta\phi}{\delta b}\Omega_b. \end{aligned} \tag{2.16}$$

By using the linear system (2.1), (2.2) and (2.3) we can write the following nine relations in terms of the eight parameters κ, τ, Ω_s, Ω_n, $div\boldsymbol{n}$,$div\boldsymbol{b}$, θ_{ns} and θ_{bs}. But we take (2.20), (2.21) and (2.22) relations for this work.

$$\frac{\delta}{\delta b}\theta_{ns}+\frac{\delta}{\delta n}\left(\Omega_n+\tau\right)=\left(div\boldsymbol{n}\right)\left(\Omega_s-2\Omega_n-2\tau\right)+\left(\theta_{bs}-\theta_{ns}\right)div\boldsymbol{b}+\kappa\Omega_s, \tag{2.17}$$

$$\frac{\delta}{\delta b}\left(\Omega_n-\Omega_s+\tau\right)+\frac{\delta}{\delta n}\theta_{bs}=div\boldsymbol{n}\left(\theta_{ns}-\theta_{bs}\right)+div\boldsymbol{b}\left(\Omega_s-2\Omega_n-2\tau\right), \tag{2.18}$$

$$\begin{aligned}\frac{\delta}{\delta b}\left(div\boldsymbol{b}\right)+\frac{\delta}{\delta n}\left(div\boldsymbol{n}\right) &= \left(\tau+\Omega_n\right)\left(\tau+\Omega_n-\Omega_s\right)-\theta_{ns}\theta_{bs}-\tau\Omega_s \\ &\quad -\left(div\boldsymbol{b}\right)^2-\left(div\boldsymbol{n}\right)^2,\end{aligned} \tag{2.19}$$

$$\frac{\delta}{\delta s}\left(\tau+\Omega_n\right)+\frac{\delta\kappa}{\delta b}=-\Omega_n\theta_{ns}-\left(2\tau+\Omega_n\right)\theta_{bs}, \tag{2.20}$$

$$\frac{\delta}{\delta s}\theta_{bs}=-\theta_{bs}^2+\kappa div\boldsymbol{n}-\Omega_n\left(\tau+\Omega_n-\Omega_s\right)+\tau\left(\tau+\Omega_n\right), \tag{2.21}$$

$$\frac{\delta}{\delta s}\left(div\boldsymbol{n}\right)-\frac{\delta\tau}{\delta b}=-\Omega_n\left(div\boldsymbol{b}\right)-\theta_{bs}\left(\kappa+div\boldsymbol{n}\right), \tag{2.22}$$

$$\frac{\delta\kappa}{\delta n}-\frac{\delta}{\delta s}\theta_{ns}=\kappa^2+\theta_{ns}^2+\left(\tau+\Omega_n\right)\left(3\tau+\Omega_n\right)-\Omega_s\left(2\tau+\Omega_n\right), \tag{2.23}$$

$$\frac{\delta}{\delta s}\left(\tau+\Omega_n-\Omega_s\right)=-\theta_{ns}\left(\Omega_n-\Omega_s\right)+\theta_{bs}\left(-2\tau-\Omega_n+\Omega_s\right)+\kappa div\boldsymbol{b}, \tag{2.24}$$

$$\frac{\delta\tau}{\delta n}+\frac{\delta}{\delta s}\left(div\boldsymbol{b}\right)=-\kappa\left(\Omega_n-\Omega_s\right)-\theta_{ns}div\boldsymbol{b}+\left(div\boldsymbol{n}\right)\left(-2\tau+\Omega_n+\Omega_s\right). \tag{2.25}$$

§3. General Properties

The relation

$$\frac{\delta\boldsymbol{n}}{\delta n}=\kappa_n\boldsymbol{n}_n=-\theta_{ns}\boldsymbol{t}-\left(div\boldsymbol{b}\right)\boldsymbol{b} \tag{3.1}$$

gives that the unit normal to the n-lines and their curvatures are given, respectively, by

$$\boldsymbol{n}_n=\frac{-\theta_{ns}\boldsymbol{t}-\left(div\boldsymbol{b}\right)\boldsymbol{b}}{\left\|-\theta_{ns}-\left(div\boldsymbol{b}\right)\boldsymbol{b}\right\|}=\frac{-\theta_{ns}\boldsymbol{t}-\left(div\boldsymbol{b}\right)\boldsymbol{b}}{-\theta_{ns}}, \tag{3.2}$$

$$\kappa_n=-\theta_{ns}. \tag{3.3}$$

In addition, from the relation (2.11) can be written,

$$curl\boldsymbol{n}=\Omega_n\boldsymbol{n}+\kappa_n\boldsymbol{b}_n, \tag{3.4}$$

where

$$\boldsymbol{b}_n=\boldsymbol{n}\times\boldsymbol{n}_n=\frac{-\left(div\boldsymbol{b}\right)(1,0,0)+\theta_{ns}\boldsymbol{b}}{-\theta_{ns}} \tag{3.5}$$

gives the unit binormal to the n-lines. Similarly, the relation

$$\frac{\delta \boldsymbol{b}}{\delta b} = \kappa_b \boldsymbol{n}_b = -\theta_{bs} \boldsymbol{t} - (div \boldsymbol{n})\, \boldsymbol{n} \tag{3.6}$$

gives that the unit normal to the b-lines and their curvature are given, respectively, by

$$\boldsymbol{n}_b = \frac{\theta_{bs} \boldsymbol{t} + (div \boldsymbol{n})\, \boldsymbol{n}}{\theta_{bs}}, \tag{3.7}$$

$$\kappa_b = -\theta_{bs}. \tag{3.8}$$

Moreover, from the relation (2.13) we can be written as

$$curl \boldsymbol{b} = \Omega_b \boldsymbol{b} + \kappa_b \boldsymbol{b}_b, \tag{3.9}$$

where

$$\boldsymbol{b}_b = \boldsymbol{b} \times \boldsymbol{n}_b = \frac{\theta_{bs} \boldsymbol{n} - (div \boldsymbol{n})\,(1,0,0)}{\theta_{bs}} \tag{3.10}$$

is the unit binormal to the b-line. To determine the torsions of the n-lines and b-lines, we take the relations

$$\frac{\delta \boldsymbol{b}_n}{\delta n} = -\tau_n \boldsymbol{n}_n, \tag{3.11}$$

$$\frac{\delta \boldsymbol{b}_b}{\delta b} = -\tau_b \boldsymbol{n}_b, \tag{3.12}$$

respectively. Thus, from (3.11) we have

$$-\frac{\delta}{\delta n}(\ln|\kappa_n|)\,(div \boldsymbol{b}) - \frac{\delta}{\delta n}(div \boldsymbol{b}) - \theta_{ns}(\Omega_b + \tau) = \tau_n \theta_{ns}, \tag{3.13}$$

$$-\frac{\delta}{\delta n}\ln|\kappa_n|\,\theta_{ns} + \frac{\delta}{\delta n}\theta_{ns} = \tau_n\,(div \boldsymbol{b}). \tag{3.14}$$

Accordingly,

$$\tau_n = \begin{cases} -(\Omega_b + \tau) + \frac{div \boldsymbol{b}}{\theta_{ns}} \frac{\delta}{\delta n} \ln\left|\frac{\theta_{ns}}{div \boldsymbol{b}}\right| & \text{if } div \boldsymbol{b} \neq 0,\ \theta_{ns} \neq 0 \\ -(\Omega_b + \tau) & \text{if } div \boldsymbol{b} = 0,\ \theta_{ns} \neq 0 \\ & \text{or } \theta_{ns} = 0,\ div \boldsymbol{b} \neq 0. \end{cases} \tag{3.15}$$

Similarly, from (3.12) we have

$$-\frac{\delta}{\delta b}(\ln \kappa_b)\,(div \boldsymbol{n}) + \frac{\delta}{\delta b}(div \boldsymbol{n}) - \theta_{bs}(\Omega_n + \tau) = \tau_b \theta_{bs}, \tag{3.16}$$

$$\frac{\delta}{\delta b}(\ln \kappa_b)\,\theta_{bs} - \frac{\delta}{\delta b}\theta_{bs} = \tau_b\,(div \boldsymbol{n}). \tag{3.17}$$

Thus,

$$\tau_b = \begin{cases} -\left(\Omega_n + \tau\right) - \frac{(div\boldsymbol{n})}{\theta_{bs}} \frac{\delta}{\delta b} \ln \left| \frac{\theta_{bs}}{div\boldsymbol{n}} \right| & \text{if } div\boldsymbol{n} \neq 0,\ \theta_{bs} \neq 0, \\ \left(\Omega_n + \tau\right) & \text{if } div\boldsymbol{n} = 0, \theta_{bs} \neq 0 \\ & \text{or } \theta_{bs} = 0,\ div\boldsymbol{n} \neq 0. \end{cases} \tag{3.18}$$

Also, we obtain an important relation

$$\Omega_s - \tau = \frac{1}{2}\left(\Omega_s + \Omega_n + \Omega_b\right) \tag{3.19}$$

is obtained by combining the equations (2.10), (2.12) and (2.14). Ω_s, Ω_n and Ω_b are defined the total moments of the $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ congruences, respectively.

In conclusion, we see that the relation (3.19) has cognate relations

$$\Omega_n - \tau_n = \frac{1}{2}\left(\Omega_n + \Omega_{n_n} + \Omega_{b_n}\right), \tag{3.20}$$

$$\Omega_b - \tau_b = \frac{1}{2}\left(\Omega_b + \Omega_{n_b} + \Omega_{b_b}\right), \tag{3.21}$$

where

$$\begin{aligned} \Omega_{n_n} &= \boldsymbol{n}_n \cdot curl\boldsymbol{n}_n, \quad \Omega_{\boldsymbol{b}_n} = \boldsymbol{b}_n \cdot curl\boldsymbol{b}_n, \\ \Omega_{n_b} &= \boldsymbol{n}_b \cdot curl\boldsymbol{n}_b, \quad \Omega_{b_b} = \boldsymbol{b}_b \cdot curl\boldsymbol{b}_b. \end{aligned} \tag{3.22}$$

§4. The Nonlinear Schrödinger Equation

In geometric restriction

$$\Omega_n = 0 \tag{4.1}$$

imposed. Here, our purpose is to obtain the nonlinear Schrodinger equation with such a restriction in G_3. The condition indicate the necessary and sufficient restriction for the existence of a normal congruence of Σ surfaces containing the s-lines and b-lines. If the s-lines and b-lines are taken as parametric curves on the member surfaces U = constant of the normal congruence, then the surface metric is given by [4]

$$I_U = ds^2 + g\left(s, b\right) db^2. \tag{4.2}$$

where $g_{11} = g(s,s), g_{12} = g(s,b), g_{22} = g(b,b)$, and

$$grad_U = \boldsymbol{t}\frac{\delta}{\delta s} + \boldsymbol{b}\frac{\delta}{\delta b} = \boldsymbol{t}\frac{\partial}{\partial s} + \frac{\boldsymbol{b}}{g^{1/2}}\frac{\partial}{\partial b}. \tag{4.3}$$

Therefore, from equation (2.1) and (2.3), we have

$$\frac{\partial}{\partial s}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{4.4}$$

$$g^{-1/2}\frac{\partial}{\partial b}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & -(\Omega_n+\tau) & \theta_{bs} \\ (\Omega_n+\tau) & 0 & div\boldsymbol{n} \\ -\theta_{bs} & -div\boldsymbol{n} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}. \tag{4.5}$$

Also, if r shows the position vector to the surface then (4.4) and (4.5) implies that

$$r_{bs} = \frac{\partial \boldsymbol{t}}{\partial b} = g^{1/2}\left[-\tau\boldsymbol{n} + \theta_{bs}\boldsymbol{b}\right] \tag{4.6}$$

and

$$r_{sb} = \frac{\partial}{\partial s}\left(g^{1/2}\boldsymbol{b}\right) = -g^{1/2}\tau\boldsymbol{n} + \frac{\partial g^{1/2}}{\partial s}\boldsymbol{b}. \tag{4.7}$$

Thus, we obtain

$$\theta_{bs} = \frac{1}{2}\frac{\partial \ln g}{\partial s}. \tag{4.8}$$

In the case $\Omega_n = 0$, the compatibility conditions equations (2.20)-(2.22) become the non-linear system

$$\frac{\partial \tau}{\partial s} + \frac{\partial \kappa}{\partial b} = -2\tau\theta_{bs}, \tag{4.9}$$

$$\frac{\partial}{\partial s}\theta_{bs} = -\theta_{bs}^2 + \kappa div\boldsymbol{n} + \tau^2, \tag{4.10}$$

$$\frac{\partial}{\partial s}(div\boldsymbol{n}) - \frac{\partial \tau}{\partial b} = -\theta_{bs}(\kappa + div\boldsymbol{n}). \tag{4.11}$$

The Gauss-Mainardi-Codazzi equations become with (4.8)

$$\frac{\partial}{\partial s}(g^{1/2}div\boldsymbol{n}) + \kappa\frac{\partial}{\partial s}(g^{1/2}) - \frac{\partial \tau}{\partial b} = 0, \tag{4.12}$$

$$\frac{\partial}{\partial s}(g\tau) + g^{1/2}\frac{\partial \kappa}{\partial b} = 0, \tag{4.13}$$

$$(g^{1/2})_{ss} = g^{1/2}(\kappa div\boldsymbol{n} + \tau^2). \tag{4.14}$$

With elimination of $div\boldsymbol{n}$ of between (4.12) and (4.14), we have

$$\frac{\partial \tau}{\partial b} = \frac{\partial}{\partial s}\left[\frac{\left(g^{1/2}\right)_{ss} - \tau^2 g^{1/2}}{\kappa}\right] + \kappa\frac{\partial}{\partial s}\left(g^{1/2}\right). \tag{4.15}$$

If we accept

$$g^{1/2} = \lambda\kappa,$$

where λ varies only in the direction normal congruence, then $\lambda b \to b$, thus the pair equations (4.13) and (4.15) reduces to

$$\kappa_b = 2\kappa_s\tau + \kappa\tau_s, \tag{4.16}$$

$$\tau_b = \left(\tau^2 - \frac{\kappa_{ss}}{\kappa} + \frac{\kappa^2}{2}\right)_s. \tag{4.17}$$

By using equations (4.16) and (4.17), we obtain

$$iq_b + q_{ss} - \frac{1}{2}\left|\langle q, q\rangle\right|^2 \bar{q} - \Phi\left(b\right) q = 0, \tag{4.18}$$

where $\Phi\left(b\right) = \left(\tau^2 - \frac{\kappa_{ss}}{\kappa} + \frac{\kappa^2}{2}\right)_{s=s_0}$. This is nonlinear Schrodinger equation of repulsive type.

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