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Some Parameters of Domination on the Neighborhood Graph

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Abstract: Let $G = (V, E)$ be a simple graph. The neighborhood graph $N(G)$ of a graph G is the graph with the vertex set $V \cup S$ where S is the set of all open neighborhood sets of G and with vertices $u, v \in V(N(G))$ adjacent if $u \in V$ and v is an open neighborhood set containing u. In this paper, we obtain the domination number, the total domination number and the independent domination number in the neighborhood graph. We also investigate these parameters of domination on the join and the corona of two neighborhood graphs.

Key Words: Neighborhood graph, domination number, Smarandachely dominating k-set, total domination, independent domination, join graph, corona graph.

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§1. Introduction

Let $G = (V, E)$ be a simple graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The neighborhood of a vertex u is denoted by $N_G(u)$ and its degree $|N_G(u)|$ by $deg_G(u)$. The minimum and maximum degree of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The open neighborhood of a set $S \subseteq G$ is the set $N(S) = \bigcup_{v \in V(G)} N(v)$, and the closed neighborhood of S is the set $N[S] = N(S) \cup S$. A cut-vertex of a graph G is any vertex $u \in V(G)$ for which induced subgraph $G \setminus \{u\}$ has more components than G. A vertex with degree 1 is called an end-vertex [1].

A dominating set is a set D of vertices of G such that every vertex outside D is dominated by some vertex of D. The domination number of G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G, and generally, a vertex set D_S^k of G is a Smarandachely dominating k-set if each vertex of G is dominated by at least k vertices of S. Clearly, if $k = 1$, such a set D_S^k is nothing else but a dominating set of G. A dominating set D is a total dominating set of G if every vertex of the graph is adjacent to at least one vertex in D. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum size of a total dominating set of G. A dominating

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set D is called an independent dominating set if D is an independent set. The independent domination number of G denoted by $\gamma_i(G)$ is the minimum size of an independent dominating set of G [1].

The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \,|\, u \in V(G_1)\, and\, v \in V(G_2)\}$. The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. For every $v \in V(G_1)$, G_2^v is the copy of G_2 whose vertices are attached one by one to the vertex v. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added [2].

We use K_n, C_n and P_n to denote a complete graph, a cycle and a path of the order n, respectively. A complete bipartite graph denotes by $K_{m,n}$ and the graph $K_{1,n}$ of order $n+1$ is a star graph with one vertex of degree n and n end-vertices.

The neighborhood graph $N(G)$ of a graph G is the graph with the vertex set $V \cup S$ where S is the set of all open neighborhood sets of G and two vertices u and v in $N(G)$ are adjacent if $u \in V$ and v is an open neighborhood set containing u. In Figure 1, a graph G and its neighborhood graph are shown. The open neighborhood sets in graph G are $N(1) = \{2, 3, 4\}$, $N(2) = \{1, 3\}$, $N(3) = \{1, 2\}$ and $N(4) = \{1\}$ [3].

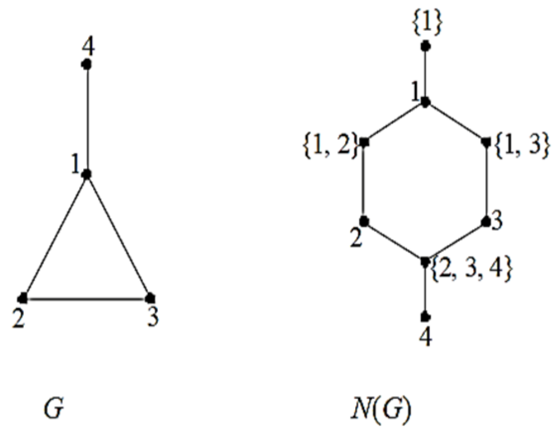

Figure 1 The graph G and the neighborhood graph of G.

In this paper, we determine the domination number, total domination number and independent domination number for the neighborhood graph of a graph G. Also, we consider the join graph and the corona graph of two neighborhood graphs and investigate some parameters of domination of these graphs.

§2. Lemma and Preliminaries

In the text follows we recall some results that establish the domination number, the total domination number and the independent domination number for graphs, that are of interest for our work.

Lemma 2.1([3]) *If G be a graph without isolated vertex of order n and the size of m, then $N(G)$ is a bipartite graph with $2n$ vertices and $2m$ edges.*

Lemma 2.2([3]) *If T be a tree with $n \geq 2$, then $N(T) = 2T$.*

Lemma 2.3 ([3]) *For a cycle C_n with $n \geq 3$ vertices,*

$$N(C_n) = \begin{cases} 2C_n & \text{if } n \text{ is even}, \\ C_{2n} & \text{if } n \text{ is odd}. \end{cases}$$

Lemma 2.4 ([3])

(*i*) *For $1 \leq m \leq n$, $N(K_{m,n}) = 2K_{m,n}$;*
(*ii*) *For $n \geq 1$, $N(\bar{K_n}) = \bar{K_n}$;*
(*iii*) *A graph G is a r-regular if and only if $N(G)$ is a r-regular graph.*

Lemma 2.5 ([1]) *Let $\gamma(G)$ be the domination number of a graph G, then*

(*i*) *For $n \geq 3$, $\gamma(C_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$;*
(*ii*) $\gamma(K_n) = \gamma(K_{1,n}) = 1$;
(*iii*) $\gamma(K_{m,n}) = 2$;
(*iv*) $\gamma(\bar{K_n}) = n$.

Lemma 2.6 ([4]) *If T be a tree of order n and l end-vertices, then*

$$\gamma(T) \geq \frac{n - l + 2}{3}.$$

Lemma 2.7 ([5]) *Let G be a r-regular graph of order n. Then*

$$\gamma(G) \geq \frac{n}{r + 1}.$$

Lemma 2.8 ([6]) *Let γ_t be the total domination number of G. Then*
(*i*) $\gamma_t(K_n) = \gamma_t(K_{n,m}) = 2$;
(*ii*) $$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \ (mod\, 4), \\ \frac{n+2}{2} & \text{if } n \equiv 2 \ (mod\, 4), \\ \frac{n+1}{2} & \text{otherwise}. \end{cases}$$
(*iiii*) *Let T be a nontrivial tree of order n and l end-vertices, then*

$$\gamma_t(T) \geq \frac{n - l + 2}{2};$$

(*iv*) *Let G be a graph , then $\gamma_t(G) \geq 1 + \frac{|C|}{2}$, where C is the set of cut-vertices of G.*

Lemma 2.9 ([7]) *Let γ_i be the independent domination number of G. Then*

(*i*) $\gamma_i(P_n) = \gamma_i(C_n) = \lceil \frac{n}{3} \rceil$;

(*ii*) $\gamma_i(K_{n,m}) = min\{n, m\}$;

(*iii*) *For a graph G with n vertices and the maximum degree Δ,*

$$\left\lceil \frac{n}{1+\Delta} \right\rceil \leq \gamma_i(G) \leq n - \Delta.$$

(*iv*) *If G is a bipartite graph of order n without isolated vertex, then*

$$\gamma_i(G) \leq \frac{n}{2};$$

(*v*) *For any tree T with n vertices and l end-vertices,*

$$\gamma_i(T) \leq \frac{n+l}{3}.$$

Lemma 2.10 ([8]) *For any graph G, $\chi(G) \leq \triangle(G) + 1$ where $\chi(G)$ is the chromatic number of G.*

Lemma 2.11 ([9]) *For any graph G, $\kappa(G) \leq \delta(G)$, where $\kappa(G)$ is the connectivity of G.*

§3. The Domination Number, the Total Domination Number and the Independent Domination Number on $N(G)$

In this section, we propose the obtained results of some parameters of domination on a neighborhood graph.

Theorem 3.1 *Let the neighborhood graph of G be $N(G)$, then*

(*i*) $\gamma(N(P_n)) = 2\lceil \frac{n}{3} \rceil$;

(*ii*) $\gamma(N(C_n)) = \begin{cases} 2\lceil \frac{n}{3} \rceil & \text{if } n \text{ is even,} \\ \lceil \frac{2n}{3} \rceil & \text{if } n \text{ is odd.} \end{cases}$

(*iii*) $\gamma(N(K_{1,n})) = \gamma(N(K_n)) = 2$;

(*iv*) *For* $2 \leq n \leq m$, $\gamma(N(K_{n,m})) = 4$;

(*v*) *For* $n \geq 2$, $\gamma(N(\bar{K_n})) = 2n$.

Proof (*i*) Using Lemma 2.2, for $n \geq 2$, $N(P_n) = 2P_n$. So, it is sufficient to consider a dominating set of P_n. By Lemma 2.5(i), $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. Therefore,

$$\gamma(N(P_n)) = 2\gamma(P_n) = 2\lceil \frac{n}{3} \rceil.$$

(*ii*) If n is even then by Lemma 2.3, $N(C_n) = 2C_n$. So, we consider a cycle C_n of order n

and using Lemma 2.5(i), we have

$$\gamma(N(C_n)) = 2\gamma(C_n) = 2\lceil\frac{n}{3}\rceil.$$

If n is odd, then since $N(C_n)$ is a cycle of order $2n$ so, $\gamma(N(C_n)) = \gamma(C_{2n}) = \lceil\frac{2n}{3}\rceil$.

The segments on (iii), (iv) and (v) can be obtained similarly by applying Lemma 2.1, Lemma 2.4 and Lemma 2.5. □

Theorem 3.2 *Let T be a tree of order n with l end-vertices. Then*

$$\frac{2}{3}(n-l+2) \leq \gamma(N(T)) \leq n.$$

Proof Using Lemma 2.2, for every tree T, $N(T) = 2T$. So, we consider a tree T to investigate its domination number. Thus, by Lemma 2.6, for every tree T of order n with l end-vertices,

$$\gamma(T) \geq \frac{n-l+2}{3}.$$

Therefore,

$$\gamma(N(T)) = 2\gamma(T) \geq 2\left(\frac{n-l+2}{3}\right).$$

Since T is without isolated vertices so, $N(T)$ is a graph without any isolated vertex. Therefore, $V(T) \subseteq V(N(T))$ is a dominating set of $N(T)$. Thus, $\gamma(N(T)) \leq n$. It completes the result. □

Theorem 3.3 *Let G be a r-regular graph. Then,*

$$\gamma(N(G)) \geq \frac{2n}{r+1}.$$

Proof Using Lemma 2.5(iii), since G is an r-regular graph so, $N(G)$ is a r-regular graph too. According to Lemma 2.1 and Lemma 2.7, we have

$$\gamma(N(G)) \geq \frac{2n}{r+1}.$$ □

Theorem 3.4 *Let $N(G)$ be a neighborhood graph of G. Then for every vertex $x \in V(G)$, $deg_G(x)$ is equal with $deg_{N(G)}(x)$.*

Proof Assume $x \in V(G)$ and $deg_G(x) = k$. So, the neighborhood set of x is $N(x) = \{y_1, \cdots, y_k\}$ where $y_i \in V(G)$. In graph $N(G)$, x is adjacent to a vertex such as $N(u)$ that consists x. Then, x is adjacent to $N(y_i)$ for every $1 \leq i \leq k$. Thus, degree of x is k in $N(G)$. Therefore, $deg_G(x) = deg_{N(G)}(x)$. □

Theorem 3.5 *Let $\gamma(N(G))$ be the domination number of $N(G)$. For any graph G of order n*

with the maximum degree $\Delta(G)$,

$$\left\lceil \frac{2n}{1+\Delta(G)} \right\rceil \leq \gamma(N(G)) \leq 2n - \Delta(G).$$

Proof Let D be a dominating set of $N(G)$. Each vertex of D can dominate at most itself and $\Delta(N(G))$ other vertices. Since by Theorem 3.4, $\Delta(N(G)) = \Delta(G)$ so,

$$\gamma(N(G)) = |D| \geq \left\lceil \frac{2n}{1+\Delta(G)} \right\rceil.$$

Now, let v be a vertex with the maximum degree $\Delta(N(G))$ and $N[v]$ be a closed neighborhood set of v in $N(G)$. Then v dominates $N[v]$ and the vertices in $V(N(G)) \setminus N[v]$ dominate themselves.

Hence, $V(N(G)) \setminus N[v]$ is the dominating set of cardinality $2n - \Delta(N(G))$. So,

$$\gamma(N(G)) \leq 2n - \Delta(N(G)) = 2n - \Delta(G). \qquad \square$$

We establish a relation between the domination number of $N(G)$ and the chromatic number $\chi(G)$ of the graph G.

Theorem 3.6 *For any graph* G,

$$\gamma(N(G)) + \chi(G) \leq 2n + 1.$$

Proof By Theorem 3.5, $\gamma(N(G)) \leq 2n - \Delta(G)$ and by Lemma 2.10, $\chi(G) \leq \Delta(G) + 1$. Thus,

$$\gamma(N(G)) + \chi(G) \leq 2n + 1. \qquad \square$$

We obtain a relation between the domination number of $N(G)$ and the connectivity $\kappa(G)$ of G following.

Theorem 3.7 *For any graph* G,

$$\gamma(N(G)) + \kappa(G) \leq 2n.$$

Proof By Theorem 3.5, $\gamma(N(G)) \leq 2n - \Delta(G)$ and by Lemma 2.11, $\kappa(G) \leq \delta(G)$. Therefore,

$$\gamma(N(G)) + \kappa(G) \leq 2n - \Delta(G) + \delta(G),$$

since, $\delta(G) \leq \Delta(G)$ so,

$$\gamma(N(G)) + \kappa(G) \leq 2n. \qquad \square$$

The following theorem is an easy consequence of the definition of $N(G)$, Lemmas 2.2–2.4

and Lemma 2.8.

Theorem 3.8 *Let the neighborhood graph of G be $N(G)$ and $\gamma_t(N(G))$ be the total domination number of $N(G)$. Then*

$$(i)\quad \gamma_t(N(P_n)) = \gamma_t(N(C_n)) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4, \\ n+2 & \text{if } n \equiv 2 \pmod 4, \\ n+1 & \text{otherwise,} \end{cases}$$

(ii) For every $n, m \geq 1$, $\gamma_t(N(K_{m,n})) = 4$;

(iii) For $n \geq 2$, $\gamma_t(N(K_n)) = 4$.

Theorem 3.9 *Let G be a graph of order n without isolated vertices and with the maximum degree Δ. Then,*

$$\gamma_t(N(G)) \geq \frac{2n}{\Delta}.$$

Proof Let D be a total dominating set of $N(G)$. Then, every vertex of $V(N(G))$ is adjacent to some vertices of D. Since, every $v \in D$ can have at most $\Delta(N(G))$ neighborhood, it follows that $\Delta(N(G))\gamma_t(N(G)) \geq |V(N(G))| = 2n$. By Theorem 3.4, $\Delta(N(G)) = \Delta(G) = \Delta$ so, $\Delta\gamma_t(N(G)) \geq 2n$. Therefore,

$$\gamma_t(N(G)) \geq \frac{2n}{\Delta}. \qquad \square$$

Theorem 3.10 *Let T be a nontrivial tree of order n and l end-vertices. Then,*

$$\gamma_t(N(T)) \geq n + 2 - l.$$

Proof Using Lemma 2.2, $N(T) = 2T$ and so, $\gamma_t(N(T)) = 2\gamma_t(T)$. By Lemma 2.8(iv),

$$\gamma_t(T) \geq \frac{n+2-l}{2}.$$

Therefore,

$$\gamma_t(N(T)) = 2\gamma_t(T) \geq 2\Big(\frac{n+2-l}{2}\Big) = n + 2 - l. \qquad \square$$

Theorem 3.11 *Let G be a graph with x cut-vertices. Then,*

$$\gamma_t(N(G)) \geq 1 + x.$$

Proof Let C be the set of cut-vertices of $N(G)$. Since for every cut-vertex u of G, u and $N(u)$ are both cut-vertices in $N(G)$ so, $|C| = 2x$. By Lemma 2.8(iv), $\gamma_t(N(G)) \geq 1 + \frac{|C|}{2}$. Therefore, we have

$$\gamma_t(N(G)) \geq 1 + \frac{|C|}{2} = 1 + \frac{2x}{2} = 1 + x. \qquad \square$$

Theorem 3.12 *Let $\gamma_i(G)$ be the independent domination number of G. Then*

(i) $\gamma_i(N(K_{n,m})) = 2m$;
(ii) $\gamma_i(N(K_{1,n})) = 2$;
(iii) $\gamma_i(N(\bar{K_n})) = 2n$;
(iv) $\gamma_i(N(P_n)) = 2\lceil\frac{n}{3}\rceil$;
(v) $\gamma_i(N(C_n)) = \begin{cases} 2\lceil\frac{n}{3}\rceil & \text{if } n \text{ is even,} \\ \lceil\frac{2n}{3}\rceil & \text{if } n \text{ is odd.} \end{cases}$

Proof The theorem easily proves using Lemma 2.3, Lemma 2.4(i, ii), Lemma 2.5 and Lemma 2.9(i, ii). □

Theorem 3.13 *For a graph G with n vertices and the maximum degree Δ,*

$$\left\lceil \frac{2n}{1+\Delta} \right\rceil \leq \gamma_i(N(G)) \leq 2n - \Delta.$$

Proof It is easy to see that $N(G)$ is a graph of order $2n$ and the maximum degree Δ. So, using Lemma 2.9(iii) we have the result. □

We establish a relation between the independent domination number of $N(G)$ and the chromatic number $\chi(G)$ of G.

Theorem 3.14 *For any graph G,*

$$\gamma_i(N(G)) + \chi(G) \leq 2n + 1.$$

Proof By Theorem 3.13, $\gamma_i(N(G)) \leq 2n - \Delta(G)$ and by Lemma 2.10, $\chi(G) \leq \Delta(G) + 1$. So,

$$\gamma_i(N(G)) + \chi(G) \leq 2n + 1.$$ □

The following theorem is the relation between the independent domination number of $N(G)$ and the connectivity $\kappa(G)$ of G.

Theorem 3.15 *For any graph G,*

$$\gamma_i(N(G)) + \kappa(G) \leq 2n.$$

Proof By Theorem 3.13, $\gamma_i(N(G)) \leq 2n - \Delta(G)$ and by Lemma 2.11, $\kappa(G) \leq \delta(G)$. So,

$$\gamma_i(N(G)) + \kappa(G) \leq 2n - \Delta(G) + \delta(G) \leq 2n.$$ □

Theorem 3.16 *Let G be a simple graph of order n and without any isolated vertex. Then*

$$\gamma_i(N(G)) \leq n.$$

Proof For every graph G with n vertices, $N(G)$ is a bipartite graph of order $2n$. Since G doesn't have any isolated vertex so, $N(G)$ is a graph without isolated vertex. Thus, by Lemma 2.9(iv) we have

$$\gamma_i(N(G)) \leq \frac{2n}{2} = n. \qquad \square$$

Theorem 3.17 *Let T be a tree with n vertices and l end-vertices without isolated vertices. Then*

$$\gamma_i(N(T)) \leq \frac{2}{3}(n+2l).$$

Proof For every tree T, $N(T) = 2T$. Let v be an end-vertex of G. Then, the corresponding vertices of v and $N(v)$ are end-vertices in $N(G)$. Thus, if T has l end-vertices then $2l$ end-vertices are in $N(T)$. So, by Lemma 2.9(v) we have

$$\gamma_i(N(T)) = 2\gamma_i(T) \leq 2\Big(\frac{n+2l}{3}\Big). \qquad \square$$

§4. The Results of the Combination of Neighborhood Graphs

In this section, we consider two graphs G_1 and G_2 and study the join and the corona of their neighborhood graphs in two cases. In Section 4.1, we consider two cases for the join of graphs: i) the neighborhood graph of G_1+G_2 that denotes by $N(G_1+G_2)$, ii) the join of two graphs $N(G_1)$ and $N(G_2)$. So, the domination number, the total domination number and the independent domination number of these graphs are obtained. In Section 4.2, we study the domination number, the total domination number and the independent domination number on two cases of the corona graphs: *i)* $N(G_1 \circ G_2)$ and *ii)* $N(G_1) \circ N(G_2)$.

4.1 The Join of Neighborhood Graphs

Let G_1 be a simple graph of order n_1 with m_1 edges and G_2 be a simple graph with n_2 vertices and m_2 edges. By the definition of the join of two graphs, $G_1 + G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2 + m_1m_2$ edges. So, the neighborhood graph of $G_1 + G_2$ has $2(n_1 + n_2)$ vertices and $2m$ edges where $m = m_1 + m_2 + m_1m_2$. For every $x \in V(G_1 + G_2)$ that $x \in V(G_1)$, we have $deg_{G_1+G_2}(x) = deg_{G_1}(x) + n_2$. Also, if $y \in V(G_1 + G_2)$ and $y \in V(G_2)$ then $deg_{G_1+G_2}(y) = deg_{G_2}(y) + n_1$. On the other hand, using Theorem 3.4 we know that $deg_G(x) = deg_{N(G)}(x)$. So, $deg_{G_1+G_2}(x) = deg_{N(G_1+G_2)}(x)$. Thus, if $x \in V(G_1)$, then $deg_{N(G_1+G_2)}(x) = deg_{G_1}(x)+n_2$ and if $y \in V(G_2)$ then $deg_{N(G_1+G_2)}(y) = deg_{G_2}(y) + n_1$.

Now, let G_1 and G_2 be simple graphs without any isolated vertex. Thus, the join of $N(G_1)$ and $N(G_2)$ denotes $N(G_1) + N(G_2)$ of order $2(n_1 + n_2)$. Also, $N(G_1 + G_2)$ has $2m_1 + 2m_2 + 4m_1m_2$ edges. Therefore, $E(N(G_1)+N(G_2)) = E(N(G_1+G_2))+2m_1m_2$. Also, we can obtain for every $x \in V(N(G_1))$, $deg_{N(G_1)+N(G_2)}(x) = deg_{N(G_1)}(x) + 2n_2$ and for every $y \in V(G_2)$, $deg_{N(G_1)+N(G_2)}(y) = deg_{N(G_2)}(y) + 2n_1$.

Theorem 4.1 *Let G_1 and G_2 be simple graphs without isolated vertex. If order of G_1 is n_1 and $\Delta(G_1) \geq n_1 - 1$, then*

$$\gamma(N(G_1 + G_2)) = \gamma_i(N(G_1 + G_2)) = 2.$$

Proof Let $x \in V(G_1)$ be a vertex with the maximum degree at least $n_1 - 1$. So, x dominates $n_1 - 1$ vertices of G_1. Let $D = \{x, N_{G_1+G_2}(x)\}$ and $N_{G_1+G_2}(x)$ be the open neighborhood set of x in $G_1 + G_2$. Since, every vertex of G_1 is adjacent to all of vertices of G_2 in $G_1 + G_2$ so, the degree of x in $G_1 + G_2$ is $n_1 + n_2 - 1$ and x dominates $n_1 + n_2 - 1$ in $N(G_1 + G_2)$. Similarity, $N_{G_1+G_2}(x)$ dominates $n_1 + n_2 - 1$ vertices of $N(G_1 + G_2)$. So, $\gamma(N(G_1 + G_2)) = |D| = 2$.

Since, x and $N_{G_1+G_2}(x)$ are not adjacent in $N(G_1 + G_2)$. Thus, D is an independent dominating set in $N(G_1 + G_2)$. Therefore, $\gamma_i(N(G_1 + G_2)) = 2$. □

Theorem 4.2 *Let G_1 and G_2 be simple graphs without isolated vertices. Then*

$$2 \leq \gamma(N(G_1 + G_2)) \leq 4.$$

Proof It is clearly to obtain $\gamma(N(G_1 + G_2)) \geq 2$. Let $S = \{x, N_{G_1+G_2}(x), y, N_{G_1+G_2}(y)\}$ where $x \in V(G_1)$ and $y \in V(G_2)$. Then, x dominates all of vertices of G_2 in $G_1 + G_2$ and so, all of vertices of $N(G_1 + G_2)$ that are the corresponding set to the neighborhoods of $V(G_2)$. Similarity, $y \in V(G_2)$ dominates n_1 vertices of $N(G_1 + G_2)$. It is shown that S is a dominating set of $N(G_1 + G_2)$. Therefore, the result holds. □

Theorem 4.3 *For graphs G_1 and G_2,*

$$\gamma_t(N(G_1 + G_2)) = 4.$$

Proof Assume $S = \{x, N_{G_1+G_2}(x), y, N_{G_1+G_2}(y)\}$ where $x \in V(G_1)$ and $y \in V(G_2)$. The vertex of $N_{G_1+G_2}(x)$ in $N(G_1 + G_2)$ is the corresponding vertex to the neighborhood of x in G_1. So, x dominates all of the vertices of G_1 and y dominates all of vertices of G_2. It is clearly to see that x is adjacent to $N_{G_1+G_2}(y)$ and y is adjacent to $N_{G_1+G_2}(x)$. Therefore, S is a total dominating set of $N(G_1 + G_2)$ and we have $\gamma_t(N(G_1 + G_2)) \leq |S| = 4$.

Let D be a total dominating set of $N(G_1 + G_2)$ that $|D| \leq 3$. We can assume that $D = \{x, y, z\}$. Thus, we have the following cases.

Case 1. If $x, y, z \in V(G_1 + G_2)$, then since $V(N(G_1 + G_2)) = V(G_1 + G_2) \cup S$ so, all of the vertices S are dominated by D where S is the set of all open neighborhood sets of $G_1 + G_2$. But, each of vertices of $V(G_1 + G_2)$ in $V(N(G_1 + G_2))$ is not dominated by D. Thus, it is a contradiction.

Case 2. Let one of vertices of D be in $V(G_1+G_2)$ and remained vertices be in S of $N(G_1+G_2)$. Without loss of generality suppose that $x \in V(G_1)$. So, $x \in V(G_1 + G_2)$ and $y, z \in S$. since x

doesn't dominate $N_{G_1+G_2}(x)$ and y, z don't dominate the corresponding vertices to y and z in $V(G_1+G_2)$ so, D is not the dominate set in $N(G_1+G_2)$. So, it is a contradiction.

Therefore, $\gamma_t(N(G_1+G_2)) \geq 4$. □

Theorem 4.4 *For graphs G_1 and G_2,*

(i) $\gamma(N(G_1)+N(G_2)) = 2$;

(ii) $\gamma_t(N(G_1)+N(G_2)) = 2$.

Proof Using the definition of the total dominating set and the structure of the join of two graphs, the result is hold. □

4.2 **The Corona of Neighborhood Graphs**

In this section, the results of the investigating of the corona on the neighborhood graphs are proposed.

Theorem 4.5 *Let G be a connected graph of order m and H any graph of order n. Then*

$$\gamma(N(G) \circ N(H)) = 2m.$$

Proof According to the definition of the corona G and H, for every $v \in N(G)$, $V(v+N(H)^v) \cap V(N(G)) = \{v\}$ in which $N(H)^v$ is copy of $N(H)$ whose vertices are attached one by one to the vertex v. Thus, $\{v\}$ is a dominating set of $v+N(H)^v$ for $v \in V(N(G))$. Therefore, $V(N(G))$ is a dominating set of $N(G) \circ N(H)$ and $\gamma(N(G) \circ N(H)) \leq 2m$.

Let D be a dominating set of $N(G) \circ N(H)$. We show that $D \cap V(v+N(H)^v)$ is a dominating set of $v+N(H)^v$ for every $v \in V(N(G))$.

If $v \in D$, then $\{v\}$ is a dominating set of $v+N(H)^v$. It follows that $V(v+N(H)^v) \cap D$ is a dominating set of $v+N(H)^v$. If $v \notin D$ and let $x \in V(v+N(H)^v) \setminus D$ with $x \neq v$. Since, D is a dominating set of $N(G) \circ N(H)$, there exists $y \in D$ such that $xy \in E(N(G) \circ N(H))$. Then, $y \in V(N(H)^v) \cap D$ and $xy \in E(v+N(H)^v)$. Therefore, it completes the result.

Since $D \cap V(v+N(H)^v)$ is a dominating set of $v+N(H)^v$ for every $v \in V(N(G))$ so, $\gamma(N(G) \circ N(H)) = |D| \geq 2m$. It completes the proof. □

Theorem 4.6 *Let G be a connected graph of order m and H any graph of order n. Then*

$$\gamma_t(N(G) \circ N(H)) = 2m.$$

Proof It is easily to obtain that $V(N(G))$ is a total dominating set for $N(G) \circ N(H)$. So, $\gamma_t(N(G) \circ N(H)) \leq 2m$.

Let D be a total dominating set of $N(G) \circ N(H)$. Then, for every $v \in V(N(G))$, $|V(v+N(H)^v) \cap D| \geq 1$. So, $\gamma_t((N(G) \circ N(H)) = |D| \geq 2m$. Therefore, $\gamma_t(N(G) \circ N(H)) = 2m$. □

Theorem 4.7 *Let G be a simple graph of order n without isolated vertex. Then*

$$\gamma_i(N(G) \circ K_1) = 2n.$$

Proof It is clearly that there exists $2n$ end-vertices in $N(G) \circ K_1$. Since, the set of these end-vertices is the dominating set and the independent set in $N(G) \circ K_1$ so, the result holds. □

Theorem 4.8 *Let G be a simple graph without isolated vertex. Then*

$$N(G \circ K_1) \cong N(G) \circ K_1.$$

Proof Two graphs are isomorphism, if there exists the function bijection between the vertex sets of these graphs. So, we consider the function $f : V(N(G \circ K_1)) \longrightarrow V(N(G) \circ K_1)$ where for every u and v in $V(N(G \circ K_1))$ if $uv \in E(N(G \circ K_1))$ then $f(u)f(v) \in E(N(G) \circ K_1)$. It means that there exists an one to one correspondence between the vertex sets and the edge sets of $N(G \circ K_1)$ and $N(G) \circ K_1$. We easily obtain the following results:

For $N(G \circ K_1)$, $|V(N(G \circ K_1))| = 2|V(G \circ K_1)| = 2(2n) = 4n$ and $|E(N(G \circ K_1))| = 2|E(G \circ K_1)| = 2(m + n)$. Also, for graph $N(G) \circ K_1$, we have

$$\begin{aligned} |V(N(G) \circ K_1)| &= 2|V(N(G))| = 4n, \\ |E(N(G) \circ K_1)| &= 2n + |E(N(G))| = 2n + 2m = 2(n + m). \end{aligned}$$

For any $x \in V(N(G \circ K_1))$ with $deg_{N(G \circ K_1)}(x) = 1$, then $x \notin V(G)$ and $x \in V(N(G))$. On the other hand, if $y \in V(N(G) \circ K_1)$ and $deg_{N(G) \circ K_1}(y) = 1$ then, $y \notin V(N(G))$. Thus, $x \in N(G \circ K_1)$ is corresponding to y in $N(G) \circ K_1$. Also, using Theorem 3.4, if $x \in V(G)$, then $deg_{N(G \circ K_1)}(x) = deg_{G \circ K_1}(x)$ and $deg_{N(G) \circ K_1}(x) = deg_{G \circ K_1}(x)$. Therefore, if $x \in V(G)$ then, the degree of x in $N(G \circ K_1)$ is equal with the degree of x in $N(G) \circ K_1$. These results are shown that there exists an one to one correspondence between two graphs $N(G) \circ K_1$ and $N(G \circ K_1)$. □

Theorem 4.8 is shown that the obtained results on some parameters of domination of two graphs $N(G \circ K_1)$ and $N(G) \circ K_1$ are equal. So, Theorems 4.5–4.7 hold for $N(G \circ K_1)$ for any graph G.

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