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Some Properties of Conformal β -Change

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Abstract: We have considered the conformal β-change of the Finsler metric given by

$$L(x,y) \to \bar{L}(x,y) = e^{\sigma(x)} f(L(x,y), \beta(x,y)),$$

where $\sigma(x)$ is a function of x, $\beta(x,y) = b_i(x)y^i$ is a 1-form on the underlying manifold M^n,and $f(L(x,y), \beta(x,y))$ is a homogeneous function of degree one in L and β.We have studied quasi-C-reducibility, C-reducibility and semi-C-reducibility of the Finsler space with this metric. We have also calculated V-curvature tensor and T-tensor of the space with this changed metric in terms of v-curvature tensor and T-tensor respectively of the space with the original metric.

Key Words: Conformal change, β-change, Finsler space, quasi-C-reducibility, C-reducibility, semi-C-reducibility, V-curvature tensor, T-tensor.

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§1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space on the differentialble manifold M^n equipped with the fundamental function L(x,y).B.N.Prasad and Bindu Kumari and C. Shibata [1,2] have studied the general case of β-change,that is, $L^*(x,y) = f(L,\beta)$,where f is positively homogeneous function of degree one in L and β, and β given by $\beta(x,y) = b_i(x)y^i$ is a one- form on M^n. The β-change of special Finsler spaces has been studied by H.S.Shukla, O.P.Pandey and Khageshwar Mandal [7].

The conformal theory of Finsler space was initiated by M.S. Knebelman [12] in 1929 and has been investigated in detail by many authors (Hashiguchi [8] ,Izumi[4,5] and Kitayama [9]). The conformal change is defined as $L^*(x,y) = e^{\sigma(x)}L(x,y)$, where $\sigma(x)$ is a function of position only and known as conformal factor. In 2008, Abed [15,16] introduced the change $\bar{L}(x,y) = e^{\sigma(x)}L(x,y) + \beta(x,y)$, which he called a β-conformal change, and in 2009 and 2010,Nabil L.Youssef, S.H.Abed and S.G. Elgendi [13,14] introduced the transformation $\bar{L}(x,y) = f(e^{\sigma}L, \beta)$, which is β-change of conformally changed Finsler metric L. They have not only established the relationships between some important tensors of (M^n, L) and the corresponding tensors of $(M^n, \bar{L})$, but have also studied several properties of this change.

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We have changed the order of combination of the above two changes in our paper [6], where we have applied β-change first and conformal change afterwards, i.e.,

$$\bar{L}(x,y) = e^{\sigma(x)} f(L(x,y), \beta(x,y)), \tag{1.1}$$

where $\sigma(x)$ is a function of x, $\beta(x,y) = b_i(x)y^i$ is a 1-form. We have called this change as conformal β-change of Finsler metric. In this paper we have investigated the condition under which a conformal β-change of Finsler metric leads a Douglas space into a Douglas space.We have also found the necessary and sufficient conditions for this change to be a projective change.

In the present paper,we investigate some properties of conformal β-change. The Finsler space equipped with the metric $\bar{L}$ given by (1.1) will be denoted by $\bar{F^n}$.Throughout the paper the quantities corresponding to $\bar{F^n}$ will be denoted by putting bar on the top of them.We shall denote the partial derivatives with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively. The Fundamental quantities of F^n are given by

$$g_{ij} = \dot{\partial}_i \dot{\partial}_j \frac{L^2}{2} = h_{ij} + l_i l_j, \quad l_i = \dot{\partial}_i L.$$

Homogeneity of f gives

$$Lf_1 + \beta f_2 = f, \tag{1.2}$$

where subscripts 1 and 2 denote the partial derivatives with respect to L and β respectively. Differentiating above equations with respect to L and β respectively, we get

$$Lf_{12} + \beta f_{22} = 0 \text{ and } Lf_{11} + \beta f_{21} = 0. \tag{1.3}$$

Hence we have

$$f_{11}/\beta^2 = (-f_{12})/L\beta = f_{22}/L^2, \tag{1.4}$$

which gives

$$f_{11} = \beta^2\omega, f_{12} = -L\beta\omega, f_{22} = L^2\omega, \tag{1.5}$$

where Weierstrass function ω is positively homogeneous of degree -3 in L and β. Therefore

$$L\omega_1 + \beta\omega_2 + 3\omega = 0, \tag{1.6}$$

where ω_1 and ω_2 are positively homogeneous of degree -4 in L and β. Throughout the paper we frequently use the above equations without quoting them. Also we have assumed that f is not linear function of L and β so that $\omega \neq 0$.

The concept of concurrent vector field has been given by Matsumoto and K. Eguchi [11] and S. Tachibana [17], which is defind as follows:

The vector field b_i is said to be a concurrent vector field if

$$b_{i|j} = -g_{ij} \qquad b_i|_j = 0, \tag{1.7}$$

where small and long solidus denote the h- and v-covariant derivatives respectively. It has been proved by Matsumoto that b_i and its contravariant components b^i are functions of coordinates alone. Therefore from the second equation of (1.7),we have $C_{ijk}b^i = 0$.

The aim of this paper is to study some special Finsler spaces arising from conformal β-change of Finsler metric,viz., quasi-C-reducible, C-reducible and semi-C-reducible Finsler spaces. Further, we shall obtain v-curvature tensor and T-tensor of this space and connect them with v-curvature tensor and T-tensor respectively of the original space.

§2. Metric Tensor and Angular Metric Tensor of $\bar{F}^n$

Differentiating equation (1.1) with respect to y^i we have

$$\bar{l}_i = e^{\sigma}(f_1 l_i + f_2 b_i). \tag{2.1}$$

Differentiating (2.1) with respect to y^j, we get

$$\bar{h}_{ij} = e^{2\sigma}\left(\frac{ff_1}{L}h_{ij} + fL^2\omega m_i m_j\right), \tag{2.2}$$

where $m_i = b_i - \dfrac{\beta}{L}L_i$.

From (2.1) and (2.2) we get the following relation between metric tensors of F^n and $\bar{F}^n$:

$$\bar{g}_{ij} = e^{2\sigma}\left[\frac{ff_1}{L}g_{ij} - \frac{p\beta}{L}l_i l_j + (fL^2\omega + f_2^2)b_i b_j + p(b_i l_j + b_j l_i)\right], \tag{2.3}$$

where $p = f_1 f_2 - f\beta L\omega$.

The contravariant components $\bar{g}^{ij}$ of the metric tensor of $\bar{F}^n$, obtainable from $\bar{g}^{ij}\bar{g}_{jk} = \delta^i_k$, are as follows:

$$\bar{g}^{ij} = e^{-2\sigma}\left[\frac{L}{ff_1}g^{ij} + \frac{pL^3}{f^3 f_1 t}\left(\frac{f\beta}{L^2} - \Delta f_2\right)l^i l^j - \frac{L^4\omega}{ff_1 t}b^i b^j - \frac{pL^2}{f^2 f_1 t}(l^i b^j + l^j b^i)\right], \tag{2.4}$$

where $l^i = g^{ij}l_j$, $b^2 = b_i b^i$, $b^i = g^{ij}b_j$, g^{ij} is the reciprocal tensor of g_{ij} of F^n, and

$$t = f_1 + L^3\omega\Delta, \Delta = b^2 - \frac{\beta^2}{L^2}. \tag{2.5}$$

$$\begin{aligned}
&(a)\ \dot{\partial}_i f = e^{\sigma}\left(\frac{f}{L}l_i + f_2 m_i\right), \quad (b)\ \dot{\partial}_i f_1 = -e^{\sigma}\beta L\omega m_i,\\
&(c)\dot{\partial}_i f_2 = e^{\sigma}L^2\omega m_i, \quad (d)\ \dot{\partial}_i p = -\beta q L m_i,\\
&(e)\ \dot{\partial}_i\omega = -\frac{3\omega}{L}l_i + \omega_2 m_i, \quad (f)\dot{\partial}_i b^2 = -2C_{..i},\\
&(g)\ \dot{\partial}_i\Delta = -2C_{..i} - \frac{2\beta}{L^2}m_i,
\end{aligned} \tag{2.6}$$

$$(a)\ \dot{\partial}_i q = -\frac{3q}{L} l_i, \quad (b)\ \dot{\partial}_i t = -2L^3\omega C_{..i} + [L^3\Delta\omega_2 - 3\beta L\omega]m_i,$$
$$(c)\ \dot{\partial}_i q = -\frac{3q}{L} l_i + (4f_2\omega_2 + 3\omega^2 L^2 + f\omega_{22})m_i. \tag{2.7}$$

§3. Cartan's C-Tensor and C-Vectors of $\bar{F}^n$

Cartan's covariant C-tensor C_{ijk} of F^n is defined by

$$\bar{C}_{ijk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2 = \dot{\partial}_k g_{ij}$$

and Cartan's C-vectors are defined as follows:

$$C_i = C_{ijk}g^{jk}, C^i = C^i_{jk}g^{jk}. \tag{3.1}$$

We shall write $C^2 = C^i C_i$. Under the conformal β-chang (1.1) we get the following relation between Cartan's C-tensors of F^n and $\bar{F}^n$:

$$\bar{C}_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{p}{2L}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{qL^2}{2}m_i m_j m_k\right]. \tag{3.2}$$

We have

$$(a)\ m_i l^i = 0,$$
$$(b)\ m_i b^i = b^2 - \frac{\beta^2}{L^2} = \Delta = b_i m^i,$$
$$(c)\ g_{ij}m^i = h_{ij}m^i = m_j. \tag{3.3}$$

From (2.1), (2.3), (2.4) and (3.2), we get

$$\begin{aligned}\bar{C}^h_{ij} \;=\; & C^h_{ij} + \frac{p}{2ff_1}(h_{ij}m^h + h^h_j m_i + h^h_i m_j) + \frac{qL^3}{2ff_1}m_j m_k m^h \\ & -\frac{L}{ft}C_{\cdot jk}n^h - \frac{pL\Delta}{2f^2 f_1 t}h_{jk}n^h - \frac{2pL + qL^4\Delta}{2f^2 f_1 t}m_j m_k n^h,\end{aligned} \tag{3.4}$$

where $n^h = fL^2\omega b^h + pl^h$ and $h^i_j = g^{il}h_{lj}, C_{.ij} = C_{rij}b^r, C_{..i} = C_{rji}b^r b^j$ and so on.

Proposition 3.1 *Let $\bar{F}^n = (M^n, \bar{L})$ be an n-dimensional Finsler space obtained from the conformal β-change of the Finsler space $F^n = (M^n, L)$, then the normalized supporting element $\bar{l}_i$, angular metric tensor $\bar{h}_{ij}$, fundamental metric tensor $\bar{g}_{ij}$ and (h)hv-torsion tensor $\bar{C}_{ijk}$ of $\bar{F}^n$ are given by (2.1), (2.2), (2.3) and (3.2), respectively.*

From (2.4),(3.1),(3.2) and (3.4) we get the following relations between the C-vectors of of F^n and $\bar{F}^n$ and their magnitudes

$$\bar{C}_i = C_i - L^3\omega C_{i..} + \mu m_i, \tag{3.5}$$

where

$$\mu = \frac{p(n+1)}{2ff_1} - \frac{3pL^3\omega\Delta}{2ff_1} + \frac{qL^3\Delta(1-L^3\omega\Delta)}{2ff_1};$$

$$\bar{C}^i = \frac{e^{-2\sigma}L}{ff_1}C^i + M^i, \tag{3.6}$$

where

$$M^i = \frac{\mu e^{-2\sigma}L}{ff_1}m^i - \frac{L^4\omega}{ff_1}C^i_{..} - \left(C_i - e^{2\sigma}L^3\omega C_{i..} + \mu\Delta\right)\left(\frac{L^3\omega}{ff_1}b^i + \frac{L}{ft}y^i\right)$$

and

$$\bar{C}^2 = \frac{e^{-2\sigma}}{p}C^2 + \lambda, \tag{3.7}$$

where

$$\begin{aligned}\lambda \;=\; & \left(\frac{e^{-2\sigma}L}{ff_1} - L^3\omega\Delta\right)\mu^2\Delta + \frac{2\mu e^{-2\sigma}L}{ff_1}C_. \\ & -(1+2\mu\Delta)\,L^3\omega + \left(1 - 3\mu + e^{2\sigma}L^2\omega ff_1 C_.\right)L^3\omega C_{...} \\ & +L^3\omega C_{..r}\left(\left(e^{4\sigma}L\omega f^2 f_1^2 C_{i..} - \mu\Delta\right)L^3\omega b^r - e^{2\sigma}L^2\omega ff_1 C^r_{..} - 2C^r\right).\end{aligned}$$

§4. Special Cases of $\bar{F}^n$

In this section, following Matsumoto [10], we shall investigate special cases of $\bar{F}^n$ which is conformally β-changed Finsler space obtained from F^n.

Definition 4.1 *A Finsler space (M^n, L) with dimension $n \geq 3$ is said to be quasi-C-reducible if the Cartan tensor C_{ijk} satisfies*

$$C_{ijk} = Q_{ij}C_k + Q_{jk}C_i + Q_{ki}C_j, \tag{4.1}$$

where Q_{ij} is a symmetric indicatory tensor.

The equation (3.2) can be put as

$$\bar{C}_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{1}{6}\pi_{(ijk)}\left\{\left(\frac{3p}{L}h_{ij} + qL^2m_im_j\right)m_k\right\}\right],$$

where $\pi_{(ijk)}$ represents cyclic permutation and sum over the indices i, j and k.

Putting the value of m_k from equation (3.5) in the above equation, we get

$$\bar{C}_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{1}{6\mu}\pi_{(ijk)}\left\{\left(\frac{3p}{L}h_{ij} + qL^2m_im_j)(\bar{C}_k - C_k + L^3\omega C_{k..}\right)\right\}\right].$$

Rearranging this equation, we get

$$\begin{aligned}\bar{C}_{ijk} &= e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk}+\frac{1}{6\mu}\pi_{(ijk)}\left\{\left(\frac{3p}{L}h_{ij}+qL^2m_im_j\right)\bar{C}_k\right\}\right.\\ &\left.+\frac{1}{6\mu}\pi_{(ijk)}\left\{\left(\frac{3p}{L}h_{ij}+qL^2m_im_j\right)\left(L^3\omega C_{k..}-C_k\right)\right\}\right].\end{aligned}$$

Further rearrangment of this equations gives

$$\bar{C}_{ijk}=\pi_{(ijk)}(\bar{H}_{ij}\bar{C}_k)+U_{ijk}, \tag{4.2}$$

where $\bar{H}_{ij}=\frac{e^{2\sigma}}{6\mu}\{(\frac{3p}{L}h_{ij}+qL^2m_im_j)$, and

$$U_{ijk}=e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk}+\frac{1}{6\mu}\pi_{(ijk)}\left\{\left(\frac{3p}{L}h_{ij}+qL^2m_im_j\right)\left(L^3\omega C_{k..}-C_k\right)\right\}\right] \tag{4.3}$$

Since $\bar{H}_{ij}$ is a symmetric and indicatory tensor,therefore from equation (4.2) we have the following theorem.

Theorem 4.1 *Conformally β-changed Finsler space $\bar{F}^n$ is quasi-C-reducible iff the tensor U_{ijk} of equation (4.3) vanishes identically.*

We obtain a generalized form of Matsumoto's result [10] as a corollary of the above theorem.

Corollary 4.1 *If F^n is Reimannian space, then the conformally β-changed Finsler space $\bar{F}^n$ is always a quasi-C-reducible Finsler space.*

Definition 4.2 *A Finsler space (M^n, L) of dimension $n \geq 3$ is called C-reducible if the Cartan tensor C_{ijk} is written in the form*

$$C_{ijk}=\frac{1}{n+1}(h_{ij}C_k+h_{ki}C_j+h_{jk}C_i). \tag{4.4}$$

Define the tensor $G_{ijk}=C_{ijk}-\frac{1}{(n+1)}(h_{ij}C_k+h_{ki}C_j+h_{jk}C_i)$. It is clear that G_{ijk} is symmetric and indicatory. Moreover, G_{ijk} vanishes iff F^n is C-reducible.

Proposition 4.1 *Under the conformal β-change(1.1), the tensor $\bar{G}_{ijk}$ associated with the space $\bar{F}^n$ has the form*

$$\bar{G}_{ijk}=e^{2\sigma}\frac{ff_1}{L}G_{ijk}+V_{ijk} \tag{4.5}$$

where

$$\begin{aligned}V_{ijk} &= \frac{1}{(n+1)}\pi_{(ijk)}\{(e^{2\sigma}(n+1)(\alpha_1h_{ij}+\alpha_2m_im_j)m_k+e^{2\sigma}\omega L^2m_im_jC_k\\ &+e^{2\sigma}L^2\omega(ff_1h_{ij}+L^3\omega m_im_j)C_{k..}\},\end{aligned} \tag{4.6}$$

$$\alpha_1=\frac{e^{2\sigma}p}{2L}-\frac{\mu ff_1e^{2\sigma}}{L(n+1)},\quad \alpha_2=\frac{e^{2\sigma}qL^2}{6}-\frac{\mu e^{2\sigma}\omega L^2}{(n+1)}.$$

From (4.5) we have the following theorem.

Theorem 4.2 *Conformally β-changed Finsler space $\bar{F}^n$ is C-reducible iff F^n is C-reducible and the tensor V_{ijk} given by (4.6) vanishes identically.*

Definition 4.3 *A Finsler space (M^n, L) of dimension $n \geq 3$ is called semi-C-reducible if the Cartan tensor C_{ijk} is expressible in the form:*

$$C_{ijk} = \frac{r}{n+1}(h_{ij}C_k + h_{ki}C_j + h_{jk}C_i) + \frac{s}{C^2}C_iC_jC_k, \tag{4.7}$$

where r and s are scalar functions such that $r + s = 1$.

Using equations (2.2), (3.5) and (3.7) in equation (3.2), we have

$$\bar{C}_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{p}{2\mu ff_1}(\bar{h}_{ij}\bar{C}_k + \bar{h}_{ki}\bar{C}_j + \bar{h}_{jk}\bar{C}_i) + \frac{\Delta L(f_1q - 3p\omega)}{2ff_1\mu t\bar{C}^2}\bar{C}_i\bar{C}_j\bar{C}_k\right].$$

If we put

$$r' = \frac{p(n+1)}{2\mu ff_1}, s' = \frac{\Delta L(f_1q - 3p\omega)}{2ff_1\mu t},$$

we find that $r' + s' = 1$ and

$$\bar{C}_{ijk} = e^{2\sigma}\left[\frac{ff_1}{L}C_{ijk} + \frac{r'}{n+1}(\bar{h}_{ij}\bar{C}_k + \bar{h}_{ki}\bar{C}_j + \bar{h}_{jk}\bar{C}_i) + \frac{s'}{\bar{C}^2}\bar{C}_i\bar{C}_j\bar{C}_k\right]. \tag{4.8}$$

From equation (4.8) we infer that $\bar{F}^n$ is semi-C- reducible iff $C_{ijk} = 0$, i.e. iff F^n is a Reimannian space. Thus we have the following theorem.

Theorem 4.3 *Conformally β-changed Finsler space $\bar{F}^n$ is semi-C-reducible iff F^n is a Riemannian space.*

§5. v-Curvature Tensor of $\bar{F}^n$

The v-curvature tensor [10] of Finsler space with fundamental function L is given by

$$S_{hijk} = C_{ijr}C^r_{hk} - C_{ikr}C^r_{hj}$$

Therefore the v-curvature tensor of conformally β-changed Finsler space $\bar{F}^n$ will be given by

$$\bar{S}_{hijk} = \bar{C}_{ijr}\bar{C}^r_{hk} - \bar{C}_{ikr}\bar{C}^r_{hj}. \tag{5.1}$$

From equations (3.2)and(3.4), we have

$$\begin{aligned}\bar{C}_{ijr}\bar{C}^r_{hk} \quad = \quad & e^{2\sigma}\left[\frac{ff_1}{L}C_{ijr}C^r_{hk} + \frac{p}{2L}(C_{ijk}m_h + C_{ijh}m_k + C_{ihk}m_j\right.\\ & +C_{hjk}m_i) + \frac{pf_1}{2Lt}(C_{.ij}h_{hk} + C_{hk}h_{ij}) - \frac{ff_1L^2\omega}{t}C_{.ij}C_{.hk}\end{aligned}$$

$$+\frac{p^2\Delta}{4fLt}h_{hk}h_{ij}+\frac{L^2(qf_1-2p\omega)}{2t}(C_{.ij}m_km_h+C_{.hk}m_im_j)$$
$$+\frac{p(p+L^3q\Delta)}{4Lft}(h_{ij}m_hm_k+h_{hk}m_im_j)+\frac{p^2}{4Lff_1}(h_{ij}m_hm_k$$
$$+h_{hk}m_im_j+h_{hj}m_im_k+h_{hi}m_jm_k+h_{jk}m_im_h+h_{ik}m_hm_j)$$
$$\left.+\frac{L^2(2pqt+(qf_1-2p\omega)(2p+L^3q\Delta))}{4ff_1t}m_im_jm_hm_k\right]. \quad (5.2)$$

We get the following relation between v-curvature tensors of (M^n, L) and $(M^n, \bar{L})$:

$$\bar{S}_{hijk}=e^{2\sigma}\left[\frac{ff_1}{L}S_{hijk}+d_{hj}d_{ik}-d_{hk}d_{ij}+E_{hk}E_{ij}-E_{hj}E_{ik}\right], \quad (5.3)$$

where

$$d_{ij}=PC_{.ij}-Qh_{ij}+Rm_im_j, \quad (5.4)$$

$$E_{ij}=Sh_{ij}+Tm_im_j, \quad (5.5)$$

$$P=L\left(\frac{s}{t}\right)^{1/2},\quad Q=\frac{pg}{2L^2\sqrt{st}}, R=\frac{L\,(2\omega p-f_1q)}{2\sqrt{st}}, S=\frac{p}{2L^2\sqrt{f\omega}},\quad T=\frac{L\,(qf_1-\omega p)}{2f_1\sqrt{f\omega}}.$$

Proposition 5.1 *The relation between v-curvature tensors of F^n and $\bar{F}^n$ is given by (5.3).*

When b_i in β is a concurrent vector field,then $C_{.ij}=0$. Therefore the value of v-curvature tensor of $\bar{F}^n$ as given by (5.3) is reduced to the extent that $d_{ij}=Rm_im_j-Qh_{ij}$.

§6. The T-Tensor T_{hijk}

The T-tensor of F^n is defined in [3] by

$$T_{hijk}=LC_{hij}\,|_k+C_{hij}l_k+C_{hik}l_j+C_{hjk}l_i+C_{ijk}l_h, \quad (6.1)$$

where

$$C_{hij}\,|_k=\dot{\partial}_kC_{hij}-C_{rij}C^r_{hk}-C_{hrj}C^r_{ik}-C_{hir}C^r_{jk}. \quad (6.2)$$

In this section we compute the T-tensor of $\bar{F}^n$, which is given by

$$\bar{T}_{hijk}=\bar{L}\bar{C}_{hij}\bar{|}_k+\bar{C}_{hij}\bar{l}_k+\bar{C}_{hik}\bar{l}_j+\bar{C}_{hjk}\bar{l}_i+\bar{C}_{ijk}\bar{l}_h, \quad (6.3)$$

where

$$\bar{C}_{hij}\bar{|}_k=\dot{\partial}_k\bar{C}_{hij}-\bar{C}_{rij}\bar{C}^r_{hk}-\bar{C}_{hrj}\bar{C}^r_{ik}-\bar{C}_{hir}\bar{C}^r_{jk}. \quad (6.4)$$

The derivatives of m_i and h_{ij} with respect to y^k are given by

$$\dot{\partial}_km_i=-\frac{\beta}{L^2}h_{ik}-\frac{1}{L}(l_im_k),\quad \dot{\partial}_kh_{ij}=2C_{ijk}-\frac{1}{L}(l_ih_{jk}+l_jh_{ki}) \quad (6.5)$$

From equations (3.2)and (6.5), we have

$$\begin{aligned}\dot{\partial}_k\bar{C}_{hij} &= e^{2\sigma}\Big[\frac{ff_1}{L}\partial_k C_{hij}+\frac{p}{L}(C_{ijk}m_h+C_{ijh}m_k+C_{ihk}m_j+C_{hjk}m_i)\\ &-\frac{p\beta}{2L^3}(h_{ij}h_{hk}+h_{hj}h_{ik}+h_{ih}h_{jk})+\frac{p}{2L^2}(h_{jk}l_hm_i+h_{hk}l_jm_i\\ &+h_{hk}l_im_j+h_{ik}l_hm_j+h_{jk}l_im_h+h_{jk}l_hm_i+h_{ij}l_hm_k+h_{hj}l_im_k\\ &+h_{ik}l_jm_k+h_{ij}l_km_h+h_{jh}l_km_i+h_{hi}l_km_j)-\frac{\beta q}{2}(h_{ij}m_hm_k\\ &+h_{hk}m_im_j+h_{hj}m_im_k+h_{hi}m_jm_k+h_{jk}m_im_h+h_{ik}m_hm_j)\\ &-\frac{qL}{2}(l_im_jm_hm_k+l_jm_im_hm_k+l_hm_im_jm_k+h_km_im_jm_h)\\ &+\frac{L^2}{2}(4f_2\omega_2+3L^2\omega^2+f\omega_{22})m_hm_im_jm_k\Big].\end{aligned} \tag{6.6}$$

Using equations (6.5) and (5.2) in equation(6.4), we get

$$\begin{aligned}\bar{C}_{hij}\bar{|}_k &= e^{2\sigma}\frac{ff_1}{L}C_{hij}\mid_k-\frac{e^{2\sigma}p}{2L}(C_{ijk}m_h+C_{ijh}m_k+C_{ihk}m_j+C_{hjk}m_i)\\ &-pe^{2\sigma}\left(\frac{2f\beta t}{4fL^3t}+\frac{L^2p\Delta}{4fL^3t}\right)(h_{ij}h_{hk}+h_{hj}h_{ik}+h_{ih}h_{jk})-e^{2\sigma}\left(\frac{\beta q}{2}\right.\\ &\left.+\frac{p^2f_1+pqf_1L^3\Delta+3p^2}{4Lff_1t}\right)(h_{ij}m_hm_k+h_{hk}m_im_j+h_{hj}m_im_k+h_{hi}m_jm_k\\ &+h_{jk}m_im_h+h_{ik}m_hm_j)-\frac{e^{2\sigma}p}{2L^2}[l_h(h_{jk}m_i+h_{ij}m_k\\ &+h_{ik}m_j)+l_j(h_{hk}m_i+h_{ik}m_{kh}+h_{ih}m_k)+l_i(h_{hk}m_j+h_{jk}m_h\\ &+h_{hj}m_k)+l_k(h_{ij}m_h+h_{jh}m_i+h_{hi}m_j)]-\frac{e^{2\sigma}qL}{2}(l_im_jm_hm_k\\ &+l_jm_im_hm_k+l_hm_im_jm_k+h_km_im_jm_h)-\frac{pf_1e^{2\sigma}}{2Lt}(C_{.ij}h_{hk}\\ &+C_{.hj}h_{ik}+C_{.hk}h_{ij}+C_{.ik}h_h+C_{.hi}h_{jk}+C_{.jk}h_{hi})+\frac{e^{2\sigma}ff_1L^2\omega}{t}(C_{.ij}C_{.hk}\\ &+C_{.hj}C_{.ik}+C_{.hi}C_{.jk})-\frac{e^{2\sigma}L^2(qf_1-2p\omega)}{2t}(C_{.ij}m_km_h\\ &+C_{.hk}m_im_j+C_{.hj}m_im_k+C_{.ik}m_jm_h\\ &+C_{.hi}m_jm_k+C_{.jk}m_hm_i)+e^{2\sigma}\Big[\frac{L^2(4f_2\omega_2+3L^2\omega^2+f\omega_{22})}{2}\\ &-\frac{3L^2(2pqt+(qf_1-2p\omega)(2p+L^3q\Delta)}{4ff_1t}\Big]m_im_jm_hm_k.\end{aligned} \tag{6.7}$$

Using equations (2.1), (3.2) and (6.6) in equation (6.3), we get the following relation between T-tensors of Finsler spaces F^n and $\bar{F}^n$:

$$\begin{aligned}\bar{T}_{hijk} &= e^{3\sigma}\Big[\frac{f^2f_1}{L^2}T_{hijk}+\frac{f(f_1f_2+f\beta L\omega)}{2L}\ (C_{ijk}m_h+C_{ijh}m_k+C_{ihk}m_j \\ &+C_{hjk}m_i)+\frac{f^2f_1L^2\omega}{t}(C_{.ij}C_{.hk}+C_{.hj}C_{.ik}+C_{.hi}C_{.jk}) \\ &-\frac{pf_1}{2Lt}(C_{.ij}h_{hk}+C_{.hj}h_{ik}+C_{.hk}h_{ij}+C_{.ik}h_h+C_{.hi}h_{jk}+C_{.jk}h_{hi}) \\ &-\frac{fL^2(qf_1-2p\omega)}{2t}(C_{.ij}m_km_h+C_{.hk}m_im_j+C_{.hj}m_im_k \\ &+C_{.ik}m_jm_h+C_{.hi}m_jm_k+C_{.jk}m_hm_i)-\frac{p(2f\beta t+L^2p\Delta)}{4L^3t}(h_{ij}h_{hk} \\ &+h_{hj}h_{ik}+h_{ih}h_{jk})-\left(\frac{p^2f_1+pqf_1L^3\Delta+3p^2}{4Lf_1t}+\frac{\beta qf}{2}-\frac{pf_2}{L}\right) \\ &(h_{ij}m_hm_k+h_{hk}m_im_j+h_{hj}m_im_k+h_{hi}m_jm_k+h_{jk}m_im_h \\ &+h_{ik}m_hm_j)+\Big[\frac{L^2(4f_2\omega_2+3L^2\omega^2+f\omega_{22})}{2}+2L^2f_2q \\ &-\ \frac{3L^2(2pqt+(qf_1-2p\omega)(2p+L^3q\Delta)}{4f_1t}\Big]m_im_jm_hm_k\Big]. \end{aligned} \tag{6.8}$$

Proposition 6.1 *The relation between T-tensors of F^n and $\bar{F}^n$ is given by (6.7).*

If bi is a concurrent vector field in F^n, then $C_{.ij}=0$. Therefore from(6.8), we have

$$\begin{aligned}\bar{T}_{hijk} &= e^{3\sigma}\Big[\frac{f^2f_1}{L^2}T_{hijk}-\frac{p(2f\beta t+L^2p\Delta)}{4L^3t}(h_{ij}h_{hk}+h_{hj}h_{ik}+h_{ih}h_{jk}) \\ &-\left(\frac{p^2f_1+pqf_1L^3\Delta+3p^2t}{4Lf_1t}+\frac{\beta qf}{2}-\frac{pf_2}{L}\right)(h_{ij}m_hm_k+h_{hk}m_im_j \\ &+h_{hj}m_im_k+h_{hi}m_jm_k+h_{jk}m_im_h+h_{ik}m_hm_j) \\ &+\Big[2L^2f_2q+\frac{L^2(4f_2\omega_2+3L^2\omega^2+f\omega_{22})}{2}+\frac{3L^2(qf_1-2p\omega)(2p+L^3q\Delta)}{4Lff_1t} \\ &-\ \frac{3L^22pqt}{4Lff_1t}\Big]m_im_jm_hm_k\Big]. \end{aligned} \tag{6.9}$$

If bi is a concurrent vector field in F^n, with vanishing T-tensor then T-tensor of F^n is given by

$$\begin{aligned}\bar{T}_{hijk} &= e^{3\sigma}\Big[-\frac{p(2f\beta t+L^2p\Delta)}{4L^3t}(h_{ij}h_{hk}+h_{hj}h_{ik}+h_{ih}h_{jk}) \\ &-\left(\frac{p^2f_1+pqf_1L^3\Delta+3p^2t}{4Lf_1t}+\frac{\beta qf}{2}-\frac{pf_2}{L}\right)(h_{ij}m_hm_k \\ &+h_{hk}m_im_j+h_{hj}m_im_k+h_{hi}m_jm_k+h_{jk}m_im_h+h_{ik}m_hm_j) \\ &+\Big[\frac{L^2(4f_2\omega_2+3L^2\omega^2+f\omega_{22})}{2}-\frac{3L^22pqt}{4Lff_1t} \\ &+\frac{3L^2(qf_1-2p\omega)(2p+L^3q\Delta)}{4Lff_1t}+2L^2f_2q\Big]m_im_jm_hm_k\Big]. \end{aligned} \tag{6.10}$$

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