Proof of the Limits of Sine and Cosine at Infinity

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Abstract

We develop a representation of complex numbers separate from the Cartesian and polar representations and define a representing functional for converting between representations. We define the derivative of a function of a complex variable with respect to each representation and then we examine the variation within the definition of the derivative. After studying the transformation law for the variation between representations of complex numbers, we will show that the new representation has special properties which allow for a modification to the transformation law for the variation which preserves, in certain cases, the definition of the derivative. We refute a common proof that the limits of sine and cosine at infinity cannot exist. We use the modified variation in the definition of the derivative to compute the limits of sine and cosine at infinity.

Contents

§1 Development of $\hat{\mathbb{C}}$

§1.1 Properties of real numbers $\mathbb{R}$

Remark 1.1.1 Faltin *et al.* [1] present the following astute observation regarding what exactly it is which constitutes a real number:

> "Few mathematical structures have undergone as many revisions or have been presented in as many guises as the real numbers. Every generation reexamines the reals in the light of its values and mathematical objectives."

With this in mind, and knowing that the sine and cosine functions predate by many centuries the axioms of a complete ordered field rooted in the work of Dedekind [2], here we will employ a geometric definition of $\mathbb{R}$ based on the Euclidean magnitude given in Euclid's Elements. We will define real numbers as cuts in the real number line, and we will first define that line. A Euclidean magnitude and a cut in a line are geometrically interchangeable concepts but presently we will consider a more general class of magnitudes than was considered by Euclid. For this reason, we do not perfectly conform to the Euclidean definition of a magnitude with what we will call a cut. Instead, we extend Euclid's conception of $\mathbb{R}$ as needed to consider a broader scope of application than was envisioned by Euclid. Indeed, in Fitzpatrick's translation of Euclid's Elements [3] it is noted that,

> "The geometrical constructions employed in the Elements are restricted to those which can be achieved using a straight-rule and a compass."

Certainly such tools are not well-suited to studying a line extending infinitely far in both directions. For this reason, we will generalize the notion of a magnitude having an interval representation as

$$(0, x) \quad ,$$

to the more general notion of a cut in an infinite line having the form

$$(-\infty, \infty) = (-\infty, x) \cup [x, \infty) \quad .$$

Definition 1.1.2 $\mathbb{R}_0$ is a subset of all real numbers

$$\mathbb{R}_0 = \{x \in \mathbb{R} \mid (\exists n \in \mathbb{N})[-n < x < n]\} \quad .$$

Here we define $\mathbb{R}_0$ as the set of all $x \in \mathbb{R}$ such that there exists an $n \in \mathbb{N}$ allowing us to write $-n < x < n$.

Definition 1.1.3 The ∞ symbol is defined as

$$\lim_{x\to 0^{\pm}} \frac{1}{x} = \pm\infty \quad .$$

Definition 1.1.4 The real number line is a 1D space extending infinitely far in both directions. It is represented in set and interval notations respectively as

$$\mathbb{R} = \{x \mid -\infty < x < \infty\} \quad , \qquad \text{and} \qquad \mathbb{R} \equiv (-\infty, \infty) \quad .$$

Definition 1.1.5 A number x is a real number if and only if it is a cut in the real number line

$$(-\infty, \infty) = (-\infty, x) \cup [x, \infty) \quad .$$

Remark 1.1.6 While it is currently popular to combine the definition of $\mathbb{R}$ directly with axioms of the operations via what are called the axioms of a complete ordered field, presently we will define $x \in \mathbb{R}$ with Definition 1.1.5 alone. In the following sections, we will axiomatize the relevant operations.

§1.2 Properties of extended real numbers $\overline{\mathbb{R}}$

Definition 1.2.1 The extended real numbers are

$$\overline{\mathbb{R}} \quad \equiv \quad \mathbb{R} \cup \{\pm\infty\} \quad .$$

Definition 1.2.2 For $n \in \mathbb{N}$ and $x_n > 0$ with x_n being a monotonic sequence, the ∞ symbol is such that if $x_n \in \mathbb{R}$, then

$$\lim_{n\to\infty} x_n = \text{diverges} \quad .$$

On the other hand, if $x_n \in \overline{\mathbb{R}}$, then

$$\lim_{n\to\infty} x_n = \infty \quad .$$

Theorem 1.2.3 *∞ has the property of additive absorption*

$$\forall b \in \mathbb{R}_0 \quad \exists \pm\infty \in \overline{\mathbb{R}} \qquad \text{s.t.} \qquad \pm\infty + b = \pm\infty \quad .$$

Proof. Definition 1.1.3 gives

$$\infty + b = \lim_{x\to 0^+} \frac{1}{x} + b = \lim_{x\to 0^+} \frac{1+bx}{x} = \text{diverges} \quad .$$

By Definition 1.2.2, the ∞ symbol is defined as limit of any positive definite monotonic sequence which diverges.

Theorem 1.2.4 *∞ has the property of multiplicative absorption such that*

$$\forall b \in \mathbb{R}_0 \quad , \quad b \neq 0 \quad \exists \pm\infty \in \overline{\mathbb{R}} \qquad \text{s.t.} \qquad b \times \pm\infty = \text{sign}(b) \times \pm\infty \quad .$$

Proof. Definition 1.1.3 gives

$$\infty \times b = \lim_{x \to 0^+} \frac{b}{x} = \text{diverges} \quad .$$

By Definition 1.2.2, the ∞ symbol is define as limit of any positive definite monotonic sequence which diverges. It follows that $-\infty$ is the limit of any negative definite monotonic sequence which diverges.

Axiom 1.2.5 ∞ does not have an additive inverse so

$$\infty - \infty = \text{undefined} \quad .$$

Axiom 1.2.6 ∞ does not have a multiplicative inverse so

$$\frac{\infty}{\infty} = \text{undefined} \quad .$$

Remark 1.2.7 Even while ∞ does not have the inverse composition properties of the real numbers, $\overline{\mathbb{R}}$ has the useful property that one may use numbers on both sides of divergent limit equations. ∞ is a special number that $\overline{\mathbb{R}}$ was conceived to accommodate.

§1.3 Properties of modified extended real numbers $\widehat{\mathbb{R}}$

Remark 1.3.1 To define $\widehat{\mathbb{R}}$, we will first extend the definition given to the ∞ symbol (Definition 1.1.3). When ∞ appears as $\widehat{\infty}$, the hat is an instruction not to write infinity as a limit. Instead, $\pm\widehat{\infty}$ should be considered qualitatively as a vector pointing to one of the endpoints of the extended real number line $\overline{\mathbb{R}}$. There is some general notion that is possible to treat real numbers as 1D vectors although they do not totally conform to the definition of what makes a vector. For instance, the dot product of two vectors is a scalar but the dot of two $x \in \mathbb{R}_0$ is another $x \in \mathbb{R}_0$. However, as an aid in understanding $\widehat{\mathbb{R}}$, it might be useful to the reader to consider $\widehat{\infty}$ as a vector pointing to the positive endpoint of the extended real line $\overline{\mathbb{R}}$ such that for $b \in \mathbb{R}_0$, $\big(\widehat{\infty} - b\big)$ is a a vector pointing to a place near that endpoint. To the extent that every $x \in \mathbb{R}_0$ (Definition 1.1.2) can be thought of loosely as a vector with magnitude $|x|$ added to the zero vector ($x \equiv \hat{0} + \hat{x}$), every $x \in \widehat{\mathbb{R}}$ can be thought of loosely as a vector of magnitude $|x|$ added to the $\widehat{\infty}$ vector. The object $\widehat{\infty}$ can be thought of loosely as a vector whose magnitude is given by Definition 1.1.3. As we will show, suppressing the limit definition of ∞ also suppresses its absorptive properties.

Axiom 1.3.2 $\widehat{\infty}$ has the property

$$\| \pm\widehat{\infty} \| = \|\infty\| \quad .$$

Definition 1.3.3 Modified extended real numbers are

$$\widehat{\mathbb{R}} \quad \equiv \quad \{\pm(\widehat{\infty} - b) \mid b \in \mathbb{R}_0,\ b > 0\} \quad .$$

Axiom 1.3.4 For any positive $a, b \in \mathbb{R}$ and any $n \in \mathbb{N}$ such that $a < b < n$, the ordering is

$$n < \widehat{\infty} - b < \widehat{\infty} - a < \infty$$
$$-\infty < -\widehat{\infty} + a < -\widehat{\infty} + b < -n \quad .$$

Remark 1.3.5 Axiom 1.3.4 separates $\widehat{\mathbb{R}}$ from the $\mathbb{R}_0$ given in Definition 1.1.2. Here we see one of the main difference between the geometric cut definition of $\mathbb{R}$ and the complete ordered field definition: presently $\mathbb{R}$ does not globally satisfy the least upper bound property though this property is retained in $\mathbb{R}_0$. (In Main Theorem 1.3.31, we will prove that all $x \in \widehat{\mathbb{R}}$ satisfy the definition of $\mathbb{R}$ given in Definition 1.1.5, *i.e.*: all modified extended real numbers are ordinary real numbers.)

Remark 1.3.6 If

$$x_n \in \mathbb{R} \quad , \quad x > 0 \quad \text{s.t.} \quad \lim_{n\to\infty} x_n = \text{diverges} \quad ,$$

then

$$x_n \in \mathbb{R} \cup \{\widehat{\infty}\} \quad , \quad x > 0 \quad \text{s.t.} \quad \lim_{n\to\infty} x_n = \widehat{\infty} \quad .$$

Theorem 1.3.7 *The $\pm\widehat{\infty}$ symbols are not vested with the property of additive absorption.*

Proof. It follows from Theorem 1.2.3 that the absorptive properties are derived from the limit definition which is suppressed by the hat. Since we will not axiomatize this property for $\pm\widehat{\infty}$, it is not vested with it.

Definition 1.3.8 If $b < 0$, then $\pm\big(\widehat{\infty} - b\big) \neq \pm\widehat{\infty}$ is an undefined quantity. (Non-equality follows from the non-absorptivity of $\widehat{\infty}$ under addition.)

Axiom 1.3.9 The $\pm\widehat{\infty}$ symbols are vested with the property of multiplicative absorption:

$$\forall b \in \mathbb{R}_0 \quad , \quad b > 0 \quad \exists \pm\widehat{\infty} \in \overline{\mathbb{R}} \qquad \text{s.t.} \qquad b \times \pm\widehat{\infty} = \pm\widehat{\infty} \times b = \pm\widehat{\infty} \quad .$$

Remark 1.3.10 Usually arithmetic is considered to be the set of four operations with equal standing: $\{+, -, \times, \div\}$. Here we take the position that $\{+, -\}$ are more fundamental than $\{\times, \div\}$ because the $\times$ operator is built on the $+$ operator where $nx = \sum_n x$. It is the notion that $+$ and $\times$ are not on an equal footing in the foundations of arithmetic that motivates us to axiomatize only the multiplicative absorptive operation while not doing so for the additive one.

Remark 1.3.11 In Remark 1.3.1, we suggested that is useful to think of $\widehat{\infty}$ as a vector-like object translating magnitudes so that they are measured from the endpoints of the extended real line $\overline{\mathbb{R}}$ rather than its origin which could be equivalently expressed as the zero vector $\hat{0}$. Now we will suggest another thinking device motivating $\widehat{\infty}$. The hat on infinity can be considered as an instruction to delay the additive absorptive operation of ∞ as long as possible within the freedom allowed by the order of operations. In kindergarten, it is often required that all expression must be simplified as much as possible but there is no such requirement in advanced mathematics. Therefore, without ever mentioning the limit definition of ∞, we can let the hat be an instruction not to simplify expressions involving $\widehat{\infty}$ by way of the additive absorptive property shown in Theorem 1.2.3. The reader should fully note that we have not added anything to infinity with the hat. The hat is merely an instruction regarding how to use the algebraic freedom which is innate to ∞.

Definition 1.3.12 The non-contradiction property is a property of $\pm\widehat{\infty}$ which requires that the hat be removed if a contradiction is obtained by the properties of the hat.

Example 1.3.13 This example describes the nature of the non-contradiction property of $\widehat{\infty}$. By delaying or suppressing additive absorption, we may define unique numbers of the form

$$x = \pm\big(\widehat{\infty} - b\big) \quad ,$$

but there are certain instances in which additive absorption is required to avoid contradictions. Consider two series

$$x_n = \sum_{k=1}^{n} k \quad , \qquad \text{and} \qquad y_n = c_0 + \sum_{k=1}^{n} k \quad ,$$

where is $c_0 \in \mathbb{R}_0$ and $c_0 < 0$. Applying the definition of ∞ (Definition 1.2.2) we obtain

$$\lim_{n\to\infty} x_n = \infty \quad , \qquad \text{and} \qquad \lim_{n\to\infty} y_n = \infty \quad .$$

We may also write y_n as

$$y_n = c_0 + x_n \quad ,$$

to take the limit as

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \big(c_0 + x_n\big) = \lim_{n\to\infty} c_0 + \lim_{n\to\infty} x_n = c_0 + \infty \quad .$$

If we use the hat, this delivers a contradiction

$$\widehat{\infty} = \widehat{\infty} + c_0 \quad .$$

Therefore, we must invoke the non-contradiction property of Definition 1.3.12 to remove the hat. Then

$$\widehat{\infty} = \widehat{\infty} + 1 \qquad \longrightarrow \qquad \infty = \infty + 1 \quad ,$$

and the contradiction is avoided by the additive absorptive property of ∞ (Theorem 1.2.3.)

Remark 1.3.14 Since $\widehat{\infty} \notin \widehat{\mathbb{R}}$, the issue demonstrated by Example 1.3.13 cannot be a problem in the analysis of $\widehat{\mathbb{R}}$. If we tried to set up some contradiction between $(\widehat{\infty} - b)$ and $(\widehat{\infty} - b) + c_0$, then contradiction would always be avoided through $b \neq b + c_0$. Although it is easy to set up contradictions with $\widehat{\infty}$ which require us to remove the hat, there is a broad class of structures which do not invoke any contradictions and **they should be studied**.

Remark 1.3.15 $\widehat{\mathbb{R}}$ numbers are such that if $x_n \in \overline{\mathbb{R}}$, then

$$\lim_{n\to\infty} x_n = \infty \quad ,$$

but if $x_n \in \widehat{\mathbb{R}}$, then

$$\lim_{n\to\infty} x_n = \text{diverges} \quad ,$$

because $\widehat{\infty} \notin \widehat{\mathbb{R}}$, as per Definition 1.3.3.

Axiom 1.3.16 The operations $\widehat{\infty} - \widehat{\infty}$ and $\widehat{\infty}/\widehat{\infty}$ are undefined.

Axiom 1.3.17 $\widehat{\mathbb{R}}$ numbers are such that

$$\widehat{\infty} - a = \widehat{\infty} - b \qquad \iff \qquad a = b \quad .$$

Axiom 1.3.18 The additive composition laws for $\widehat{\mathbb{R}} + \mathbb{R}_0$ are commutative and associative. They take the form

$$\big(\widehat{\infty} - b\big) + a = \begin{cases} \widehat{\infty} - \big(b - a\big) & \big(b - a\big) > 0 \\ \text{undefined} & \big(b - a\big) \leq 0 \end{cases}$$
$$-\big(\widehat{\infty} - b\big) + a = \begin{cases} -\widehat{\infty} + \big(b + a\big) & \big(b + a\big) > 0 \\ \text{undefined} & \big(b + a\big) \leq 0 \end{cases} \quad .$$

Axiom 1.3.19 The additive composition laws for $\widehat{\mathbb{R}}+\widehat{\mathbb{R}}$ are commutative but not associative. They take the form

$$\begin{aligned}\pm\big(\widehat{\infty}-a\big)\pm\big(\widehat{\infty}-b\big)&=\pm\widehat{\infty}\mp\big(b+a\big)\\ \pm\big(\widehat{\infty}-a\big)\mp\big(\widehat{\infty}-b\big)&=\pm\big(b-a\big)\quad .\end{aligned}$$

Remark 1.3.20 Axiom 1.3.9 endows $\widehat{\infty}$ with multiplicative absorption. This property implies that $\pm\widehat{\infty}$ is an additive identity element of positive and negative $\widehat{\mathbb{R}}$ numbers respectively. Using the definition of multiplication that $x+x=2x$, it follows that $\pm\widehat{\infty}$ has additive identity properties because

$$\pm\big(\widehat{\infty}-b\big)\pm\widehat{\infty}=\pm\widehat{\infty}\mp b\pm\widehat{\infty}=\pm2\widehat{\infty}\mp b=\pm\big(\widehat{\infty}-b\big)\quad .$$

We have given $x\in\widehat{\mathbb{R}}$ two different additive elements. If we defined 0 as an additive identity element of $\pm\widehat{\infty}$, then we could use Axiom 1.3.19 to obtain a contradiction of the form

$$\begin{aligned}\big(\widehat{\infty}-b\big)+\widehat{\infty}&=\big(\widehat{\infty}-b\big)+0\\ -\big(\widehat{\infty}-b\big)+\big(\widehat{\infty}-b\big)+\widehat{\infty}&=-\big(\widehat{\infty}-b\big)+\big(\widehat{\infty}-b\big)+0\\ 0+\widehat{\infty}&=0+0\end{aligned}$$

By depriving $\widehat{\infty}$ of an additive identity element, we have avoided this contradiction because $0+\widehat{\infty}$ is undefined.

Example 1.3.21 The example demonstrates why $\widehat{\mathbb{R}}+\widehat{\mathbb{R}}$ cannot have the associative property. Through associativity we may easily derive a contradiction

$$\begin{aligned}\big(\widehat{\infty}-b\big)+\big(\widehat{\infty}-a\big)&=\widehat{\infty}-\big(b+a\big)\\ \Big[\big(\widehat{\infty}-b\big)+\big(\widehat{\infty}-a\big)\Big]-\big(\widehat{\infty}-a\big)&=\widehat{\infty}-\big(b+a\big)-\big(\widehat{\infty}-a\big)\\ \big(\widehat{\infty}-b\big)+\Big[\big(\widehat{\infty}-a\big)-\big(\widehat{\infty}-a\big)\Big]&=\widehat{\infty}-\big(b+a\big)-\big(\widehat{\infty}-a\big)\\ \big(\widehat{\infty}-b\big)&=-b\quad .\end{aligned}$$

When associativity is not granted, this contradiction cannot be obtained.

Axiom 1.3.22 The additive composition laws for $\widehat{\mathbb{R}}\pm\widehat{\infty}$ are commutative but not associative. They take the form

$$\begin{aligned}\pm\big(\widehat{\infty}-b\big)\pm\widehat{\infty}&=\pm\big(\widehat{\infty}-b\big)\\ \pm\big(\widehat{\infty}-b\big)\mp\widehat{\infty}&=\mp b\quad .\end{aligned}$$

Remark 1.3.23 We have used the multiplicative absorptive property of $\widehat{\infty}$ to define $\pm\widehat{\infty}\notin\widehat{\mathbb{R}}$ as a second additive identity element for positive and negative $\widehat{\mathbb{R}}$ numbers respectively.

Theorem 1.3.24 *The additive composition laws for $\widehat{\mathbb{R}}$ do not require an additive inverse for $\widehat{\infty}$ (Axiom 1.3.16.)*

Proof. Consider Axiom 1.3.19 which gives the additive composition of two $\widehat{\mathbb{R}}$ numbers

$$x_1 = \widehat{\infty} - b_1 \quad , \qquad \text{and} \qquad x_2 = -\widehat{\infty} - b_2 \quad ,$$

as

$$\big(\widehat{\infty} - b_1\big) - \big(\widehat{\infty} - b_2\big) = b_2 - b_1 \quad .$$

The case of $b_1 = b_2 = 0$ is ruled out by the definition of $\widehat{\mathbb{R}}$ because Definition 1.3.3 gives

$$\widehat{\mathbb{R}} \quad \equiv \quad \{\pm(\widehat{\infty} - b) \mid b \in \mathbb{R}_0,\ b > 0\} \quad .$$

Therefore, it is impossible to use Axiom 1.3.19 to make a statement of the form $\widehat{\infty} - \widehat{\infty} = 0$.

Theorem 1.3.25 *All $\widehat{\mathbb{R}}$ numbers have an additive inverse.*

Proof. Consider the case of $b = -a$ in Axiom 1.3.19:

$$\big(\widehat{\infty} - a\big) - \big(\widehat{\infty} - b\big) = b - a \quad .$$

Then

$$\forall x = \widehat{\infty} - a \quad \exists x' = -\big(\widehat{\infty} - a\big) \quad \text{s.t.} \quad x + x' = 0 \quad .$$

This is the definition of the additive inverse.

Axiom 1.3.26 The additive operations of $\widehat{\infty} + \mathbb{R}_0$ are commutative and associative.

Axiom 1.3.27 The arithmetic operations of $\widehat{\infty}$ with 0 are

$$\begin{aligned}
\pm\widehat{\infty} + 0 = 0 \pm \widehat{\infty} = \pm\widehat{\infty} - 0 &= \text{undefined} \\
\pm\widehat{\infty} \cdot 0 = 0 \cdot \pm\widehat{\infty} &= \text{undefined} \\
\frac{\pm\widehat{\infty}}{0} &= \text{undefined} \\
\frac{0}{\pm\,\widehat{\infty}} &= 0 \quad .
\end{aligned}$$

Example 1.3.28 Axiom 1.3.27 states that infinity does not have an additive identity element. An example motivating this condition is given by the limit

$$\lim_{x\to\infty} \big(x^2 - x\big) = \infty$$

$$\lim_{x\to\infty}\big(x^2-x\big)=\lim_{x\to\infty}x^2-\lim_{x\to\infty}x=\infty-\infty \quad .$$

This is a typical example used to demonstrate the lack of an additive inverse for ∞. If infinity is bestowed with an additive inverse then we obtain a contradiction $\infty=0$. The expression $\infty-\infty$, thus, is undefined. If we supposed that $\infty-\infty$ was defined, and then we added the hats to infinity then we could insert the additive identity to write

$$\infty=\widehat{\infty}-\widehat{\infty}=\widehat{\infty}-\widehat{\infty}+0=\widehat{\infty}-\widehat{\infty}+1-1=\big(\widehat{\infty}-1\big)-\big(\widehat{\infty}-1\big)=0 \quad .$$

(Final step follows from Theorem 1.3.25.) Through the contradiction $\infty=0$, we see that $\widehat{\infty}$ cannot have zero as an additive identity. Unhatted infinity likewise cannot have zero as an additive identity because we could write

$$\infty=\infty-\infty=\infty-\infty+\big(1-1\big)=\widehat{\infty}-\widehat{\infty}+1-1=\big(\widehat{\infty}-1\big)-\big(\widehat{\infty}-1\big)=0 \quad .$$

Here, we have simply used the freedom afforded to the order of algebraic operations not to do the additive absorptive operations before inserting the hats to make use of Theorem 1.3.25.

Remark 1.3.29 The expressions ∞ and $\widehat{\infty}$ are perfectly well defined but $\infty+0$ and $\widehat{\infty}+0$ are examples of an undefined composition. Since ∞ is not an $\widehat{\mathbb{R}}$ number, this property cannot create problems for the algebra of $\widehat{\mathbb{R}}$ numbers. Essentially, we have traded the zero additive identity element of infinity for the freedom to add and subtract $\widehat{\mathbb{R}}$ numbers.

Definition 1.3.30 Since we have not defined a composition law for $\widehat{\infty}+b$, expressions of the form $\widehat{\infty}+b=b+\widehat{\infty}$ are defined by commutative self-identity.

Main Theorem 1.3.31 *All $x\in\widehat{\mathbb{R}}$ are also $x\in\mathbb{R}$.*

Proof. By Axiom 1.3.4, $\widehat{\mathbb{R}}$ is defined such that we have for any $a,b>0$ with $a,b\in\mathbb{R}_0$:

$$\big(\widehat{\infty}-a\big)>\big(\widehat{\infty}-b\big)\qquad\iff\qquad a<b \quad .$$

Let η be an infinitesimal so that taking the limit $a\to\eta$ yields

$$\widehat{\infty}-\eta>\big(\widehat{\infty}-b\big) \quad .$$

Since η is an infinitesimal, it follows from Axiom 1.3.2 that $\|\widehat{\infty}-\eta\|=\|\infty\|$. Therefore,

$$\infty>\big(\widehat{\infty}-b\big) \quad .$$

and we have shown that numbers of the form

$$x=\pm\big(\widehat{\infty}-b\big) \quad ,\qquad \text{with}\qquad b\in\mathbb{R}_0 \quad ,\quad b>0 \quad ,$$

are less than infinity. It is obvious that they are greater than minus infinity so, therefore,

$$(-\infty,\infty)=(-\infty,\pm\widehat{\infty}\mp b)\cup[\pm\widehat{\infty}\mp b,\infty) \quad .$$

We have shown that all $x\in\widehat{\mathbb{R}}$ satisfy the definition of $\mathbb{R}$ (Definition 1.1.5.)

§1.4 Properties of modified extended complex numbers $\widehat{\mathbb{C}}$

Definition 1.4.1 Complex numbers are

$$\mathbb{C} \quad \equiv \quad \{x+iy \mid x,y\in\mathbb{R}\} \quad .$$

Definition 1.4.2 Define a class of complex numbers

$$\mathbb{C}_0 \quad \equiv \quad \{x+iy \mid x,y\in\mathbb{R}_0\} \quad .$$

Axiom 1.4.3 As ∞ and $\widehat{\infty}$ do not absorb -1, they do not absorb $\pm i$. We have four distinct compound symbols $\{\pm\widehat{\infty},\pm i\,\widehat{\infty}\}$ when positive real infinity is written as $+\widehat{\infty}$.

Definition 1.4.4 Extended complex numbers are

$$\overline{\mathbb{C}} \quad \equiv \quad \{x+iy \mid x,y\in\overline{\mathbb{R}}\} \quad .$$

Definition 1.4.5 Modified extended complex numbers are

$$\widehat{\mathbb{C}} \quad \equiv \quad \{x+iy \mid x,y\in\widehat{\mathbb{R}}\} \ .$$

Remark 1.4.6 The arithmetic properties of $\widehat{\mathbb{C}}$ are implicit in the arithmetic axioms of $\mathbb{R}_0$, $\widehat{\mathbb{R}}$, and $\widehat{\infty}$.

Corollary 1.4.7 $\widehat{\mathbb{C}}$ *is the complement of* $\mathbb{C}$ *on the Riemann sphere* $\mathbb{S}^2$.

Proof. A is the complement of B on $\mathbb{S}^2$ if and only if

$$\mathbb{S}^2 \quad \equiv \quad A\cup B \quad , \qquad \text{and} \qquad A\cap B=0 \quad .$$

The Riemann sphere is obtained from $\mathbb{C}$ by adding a point for infinity to both ends of the real and imaginary axes and then imposing conditions

$$\pm\infty=\infty \quad , \qquad \text{and} \qquad \pm\, i\infty=\infty \quad .$$

When we say, “on the Riemann sphere,” we mean the limit where $\pm\widehat{\infty}\to\infty$ and $\pm i\,\widehat{\infty}\to\infty$. Considering the usual association between the Riemann sphere and

$$\overline{\mathbb{C}} \quad \equiv \quad \{x+iy \mid x,y\in\overline{\mathbb{R}}\} \quad .$$

it follows that

$$\mathbb{S}^2 \quad \equiv \quad \{\overline{\mathbb{C}} \mid \ \pm\infty \to \infty, \ \ \pm i\infty \to \infty\} \ \ .$$

If we identify the four unique infinities of

$$\widehat{\mathbb{C}} \quad \equiv \quad \{x + iy \mid x, y \in \widehat{\mathbb{R}}\} \ .$$

as the single infinity ∞, then

$$\{\widehat{\mathbb{C}} \mid \ \pm\widehat{\infty} \to \infty, \ \ \pm i\,\widehat{\infty} \to \infty\} \quad \equiv \quad \{\infty\} \ \ .$$

Both the x and the y in any $z \in \hat{\mathbb{C}}$ depend on $\widehat{\infty}$ so both of them become ∞ under the Riemann sphere identification. Every $z \in \widehat{\mathbb{C}}$, therefore, becomes ∞ under this condition. Since it is the definition of the Riemann sphere that

$$\mathbb{S}^2 \quad \equiv \quad \mathbb{C} \cup \{\infty\} \ \ ,$$

we can use the definition of the complement to write

$$\mathbb{S}^2 \quad \equiv \quad \mathbb{C} \cup \{\widehat{\mathbb{C}} \mid \ \pm\widehat{\infty} \to \infty, \ \ \pm\, i\,\widehat{\infty} \to \infty\} \ \ , \qquad \text{and} \qquad \mathbb{C} \cap \{\infty\} = 0 \ \ .$$

This proves that $\widehat{\mathbb{C}}$ is the complement of $\mathbb{C}$ on the Riemann sphere.

§1.5 Properties of modified complex numbers $\hat{\mathbb{C}}$

Definition 1.5.1 Modified complex numbers $\hat{\mathbb{C}}$ shall be such that for every $z' \in \mathbb{C}_0$ of the form

$$z' = x + iy \ \ ,$$

there is a corresponding number $z \in \hat{\mathbb{C}}$. Modified complex numbers are defined as

$$z = \begin{cases} x + iy^+ & \text{if} \quad y > 0 \\ x & \text{if} \quad y = 0 \\ x - iy^- & \text{if} \quad y < 0 \end{cases} \ \ , \qquad \text{where} \qquad y^\pm(y) : \mathbb{R}_0 \to \widehat{\mathbb{R}} \ \ ,$$

with

$$y^+(y) = \widehat{\infty} - y \ \ , \qquad \text{and} \qquad y^-(y) = \widehat{\infty} + y \ \ .$$

Theorem 1.5.2 *For any $z \in \hat{\mathbb{C}}$, $z = x \pm iy^\pm$ implies $0 < y^\pm < \infty$.*

<u>*Proof.*</u> Axiom 1.3.2 gives

$$||\widehat{\infty}|| = \infty \ \ .$$

$y^\pm$ are such that

$$y^\pm : \mathbb{R}_0 \to \widehat{\mathbb{R}} \ \ , \qquad \text{and} \qquad y^\pm = \big(\widehat{\infty} \mp y\big) \ \ , \ \ y \in \mathbb{R}_0 \ \ .$$

By the definition of $\hat{\mathbb{C}}$, $y^\pm = \widehat{\infty}$ and $y^\pm = 0$ are not allowed. For any $a, b \in \mathbb{R}_0$ with $a, b > 0$ we have from Axiom 1.3.4

$$\big(\widehat{\infty} - a\big) > \big(\widehat{\infty} - b\big) \qquad \iff \qquad a < b \ \ ,$$

wherein $a, b > 0$. This positivity condition is required by the restriction of the domain of $y^\pm(y)$ in $z = x + iy^\pm$ (Definition 1.5.1). This shows that $y^\pm$ increase as $||y||$ decreases. Therefore,

$$\sup y^\pm = y^\pm(\inf ||y||) \ \ .$$

$y \in \mathbb{R}_0$ gives

$$\inf ||y|| = 0 \qquad \Longrightarrow \qquad \sup y^\pm = \widehat{\infty} \mp 0 = \widehat{\infty} \ \ .$$

Now we have shown that $y^\pm \in \widehat{\mathbb{R}}$ where $\widehat{\mathbb{R}}$ has the property $\widehat{\infty} \notin \widehat{\mathbb{R}}$. That all $y^\pm$ are less than $\widehat{\infty}$ follows from the definition of the supremum. To show that $y^\pm$ are always greater than zero, consider that

$$\forall b \in \mathbb{R} \ \ , \ \ \widehat{\infty} > b \qquad \Longrightarrow \qquad \widehat{\infty} - b > 0 \ \ .$$

We have proven that $0 < y^\pm < \infty$.

Alternatively, proof follows from Main Theorem 1.3.31 because

$$y^\pm \in \mathbb{R} \qquad \iff \qquad 0 < y^\pm < \infty \ \ .$$

§2 Properties of $\mathbb{C}$

§2.1 Definition of a representation of complex numbers $\mathbb{C}$

Remark 2.1.1 In this section, we will define a "representation of $\mathbb{C}$." Since the usual understanding of $\mathbb{C}$ is what we have presently called $\mathbb{C}_0$ in Definition 1.4.2, we are technically defining "a representation of $\mathbb{C}_0$." However, we will call this a representation of $\mathbb{C}$ and it should be obvious from the context what is what.

Remark 2.1.2 The reader is likely familiar with the Cartesian and polar representations of complex numbers and is accustomed to using the symbol $\mathbb{C}$ for both of them. When we use

$$z \in \mathbb{C} \qquad \Longrightarrow \qquad z = re^{i\theta} \ \ , \ \ r, \theta \in \mathbb{R} \ \ ,$$

where

$$r(x, y) = \sqrt{x^2 + y^2} \ \ , \qquad \text{and} \qquad \theta(x, y) = \tan^{-1}\left(\frac{y}{x}\right) \ \ ,$$

we do not need to define an entire new class of complex numbers with some variant of $\mathbb{C}$, call it $\mathbb{C}'$, to distinguish it from

$$z \in \mathbb{C} \quad \Longrightarrow \quad z = x + iy \ \ , \quad x, y \in \mathbb{R} \ \ .$$

Usually we use the symbol $\mathbb{C}$ to refer to both the Cartesian and polar representations. For $\overline{\mathbb{C}}$ (Definition 1.4.4), we have added ∞ to $\mathbb{C}$ so a unique symbol specifying the addition is called for. In $\widehat{\mathbb{C}}$ (Definition 1.4.5), we did not add the point at infinity to $\mathbb{C}$ but we did take away the points along the real and imaginary axes of $\mathbb{C}$ because $x, y \in \widehat{\mathbb{R}}$ implies $x \neq 0$ and $y \neq 0$. Therefore, a unique construction requires a unique label. With regards to $\hat{\mathbb{C}}$ (Definition 1.5.1), however, we have neither added the point at infinity nor taken away any points so there is a strong argument to be made that

$$\hat{\mathbb{C}} \quad \equiv \quad \mathbb{C} \ \ .$$

Definition 2.1.3 The double parentheses notation $(\!(a, b)\!)$ refers to the representation of $\mathbb{C}$ in the variables a and b.

Definition 2.1.4 $(\!(x, y)\!)$ is the Cartesian representation of $\mathbb{C}$ in which

$$z(x, y) = x + iy \ \ .$$

We say

$$(\!(x, y)\!) \quad \equiv \quad z(x, y) \quad \equiv \quad x + iy \ \ .$$

$z(x, y)$ is a complex-valued function of two real variables. We call $x + iy$ the analytical form of the Cartesian representation of $\mathbb{C}$.

Definition 2.1.5 $(\!(x_2, y_2)\!)$ is a representation of $\mathbb{C}$ if and only if $(\!(x_1, y_1)\!)$ is a representation of $\mathbb{C}$ and there exist two conversion functions

$$x_2 = x_2(x_1, y_1) \ \ , \qquad \text{and} \qquad y_2 = y_2(x_1, y_1) \ \ ,$$

whose domains are all of $\mathbb{C}_0$ or all of $\hat{\mathbb{C}}$.

Theorem 2.1.6 *$(\!(r, \theta)\!)$ is a representation of* $\mathbb{C}$.

<u>*Proof.*</u> $(\!(x, y)\!)$ is a representation of $\mathbb{C}$ and we have two conversion functions

$$r(x, y) = \sqrt{x^2 + y^2} \ \ ,$$

and

$$\theta(x, y) = \begin{cases} \tan^{-1}\left(\dfrac{y}{x}\right) & \text{if} \quad x \neq 0 \\ \dfrac{\pi}{2} & \text{if} \quad x = 0, \ y > 0 \\ -\dfrac{\pi}{2} & \text{if} \quad x = 0, \ y < 0 \end{cases} \ \ .$$

$(\!(r,\theta)\!)$ is a representation of $\mathbb{C}$ because all of $\mathbb{C}_0$ is in the domain of the conversion functions. (The case of $x = y = 0$ is covered by $r = 0$ and does not require a specification of θ.)

Remark 2.1.7 Due to the freedom to choose the sign of $\sqrt{y^2}$ we might add a rule to the definition of $\theta(x, y)$ to be more explicit. However, we will not do so and it suffices to concede that some fine nuance remains unspecified.

Definition 2.1.8 If $(\!(x_1, y_1)\!)$ and $(\!(x_2, y_2)\!)$ are two representations of $\mathbb{C}$, then there exists a representing functional of two conversion functions

$$z_{(\!(x_2,y_2)\!)}[(\!(x_1, y_1)\!)] : (\!(x_1, y_1)\!) \to (\!(x_2, y_2)\!) \quad ,$$

where the two conversion functions are

$$x_1(x_2, y_2) : (\mathbb{R}, \mathbb{R}) \to \mathbb{R} \quad , \qquad \text{and} \qquad y_1(x_2, y_2) : (\mathbb{R}, \mathbb{R}) \to \mathbb{R} \quad .$$

Remark 2.1.9 The purpose of the representing functional is to convert the analytical form of one representation into the analytical form of another. The representing functional $z_{(\!(x_2,y_2)\!)}[(\!(x_1, y_1)\!)]$ has two conversion functions in its domain $(\!(x_1, y_1)\!)$, each a function of two real variables: $x_1(x_2, y_2)$ and $y_1(x_2, y_2)$.

Definition 2.1.10 For a representing functional $z_B[A]$ as in Definition 2.1.8, A is the known analytic representation and B is the analytical form of the representation into which we will convert.

Remark 2.1.11 For a representing functional $z_B[A]$, the conversion functions must be written in the form $x_A(x_B, y_B)$ and $y_A(x_B, y_B)$. This remark will be formalized in Definition 2.1.12 but we deem it important to emphasize the distinction between a set of conversion functions and their inverse conversion function, in this case $x_B(x_A, y_A)$ and $y_B(x_A, y_A)$.

Definition 2.1.12 The rules for constructing the representing functional of the form $z_{(\!(x_2,y_2)\!)}[(\!(x_1, y_1)\!)]$ with the conversion functions $x_1(x_2, y_2)$ and $y_1(x_2, y_2)$ are:

- Choose a representing functional

$$z_{(\!(x_2,y_2)\!)}[(\!(x_1, y_1)\!)]$$

- Replace the known representation $(\!(x_1, y_1)\!)$ with its analytical form $z(x_1, y_1)$

$$z_{(\!(x_2,y_2)\!)}[z(x_1, y_1)]$$

- Replace the known real variables x_1 and y_1 with their conversion functions

$$z_{(\!(x_2,y_2)\!)}[z(x_1(x_2,y_2),y_1(x_2,y_2))]$$

- Simplify in terms of x_2 and y_2 to get the analytical form of

$$(\!(x_2,y_2)\!)$$

Example 2.1.13 Here we use the representing functional

$$z_{(\!(r,\theta)\!)}[(\!(x,y)\!)] = (\!(r,\theta)\!) \quad ,$$

to construct the polar representation of $\mathbb{C}$ from its Cartesian representation. The conversion functions are

$$x(r,\theta) = r\cos(\theta) \quad , \qquad \text{and} \qquad y(r,\theta) = r\sin(\theta) \quad .$$

The representing functional is

$$z_{(\!(r,\theta)\!)}[(\!(x,y)\!)] = z_{(\!(r,\theta)\!)}[x+iy] = r\cos(\theta) + ir\sin(\theta) = re^{i\theta} \quad .$$

Therefore,

$$(\!(r,\theta)\!) = re^{i\theta} \quad .$$

Example 2.1.14 Here we use the representing functional

$$z_{(\!(x,y)\!)}[(\!(r,\theta)\!)] = (\!(x,y)\!) \quad .$$

to construct the Cartesian representation of $\mathbb{C}$ from its polar representation. The conversion functions are

$$r(x,y) = \sqrt{x^2+y^2} \quad , \qquad \text{and} \qquad \theta(x,y) = \tan^{-1}\left(\frac{y}{x}\right) \quad .$$

For $x \neq 0$, the representing functional is

$$\begin{aligned} z_{(\!(x,y)\!)}[(\!(r,\theta)\!)] &= z_{(\!(x,y)\!)}[re^{i\theta}] \\ &= \sqrt{x^2+y^2}\, e^{i\tan^{-1}(y/x)} \\ &= \sqrt{x^2+y^2}\,\cos\left(\tan^{-1}\left(\frac{y}{x}\right)\right) + i\sqrt{x^2+y^2}\,\sin\left(\tan^{-1}\left(\frac{y}{x}\right)\right) \\ &= \sqrt{x^2+y^2}\left(\frac{1}{\sqrt{\left(\frac{y}{x}\right)^2+1}}\right) + i\sqrt{x^2+y^2}\left(\frac{\left(\frac{y}{x}\right)}{\sqrt{\left(\frac{y}{x}\right)^2+1}}\right) \\ &= x+iy \quad . \end{aligned}$$

For $x = 0$, the representing functional is

$$z_{(\!(x,y)\!)}[(\!(r,\theta)\!)] = z_{(\!(x,y)\!)}[re^{i\theta}] = \sqrt{x^2+y^2}\, e^{\pm i\pi/2} = \pm i\sqrt{y^2} \quad .$$

Within the freedom to choose the positive or negative root of y^2 we take

$$\pm i\sqrt{y^2} = iy \quad .$$

Therefore,

$$(\!(x,y)\!) = x + iy \quad .$$

Remark 2.1.15 The polar representation relies on incorporation of the number e so we should consider other representations that include different numbers such as $\widehat{\infty}$, as in Definition 1.5.1.

Definition 2.1.16 If we have a representation

$$(\!(f(x_2), g(y_2))\!) = z(x_2, y_2) \quad ,$$

then the rules for constructing

$$z_{(\!(f(x_2),g(y_2))\!)}[(\!(x_1,y_1)\!)] = (\!(f(x_2), g(y_2))\!) \quad ,$$

are:

- Choose a representing functional

$$z_{(\!(f(x_2),g(y_2))\!)}[(\!(x_1,y_1)\!)]$$

- Replace the known representation $(\!(x_1,y_1)\!)$ with its analytical form

$$z_{(\!(f(x_2),g(y_2))\!)}[z(x_1,y_1)]$$

- Map the x_1 and y_1 variables through the functions $f(x)$ and $g(x)$ respectively

$$z_{(\!(f(x_2),g(y_2))\!)}[z(f(x_1), g(y_1))]$$

- Replace the known real variables x_1 and y_1 with their conversion functions

$$z_{(\!(f(x_2),g(y_2))\!)}[z(f(x_1(x_2,y_2)), g(y_1(x_2,y_2)))]$$

- Simplify in terms of x_2 and y_2 to get the analytical form of

$$(\!(f(x_2), g(y_2))\!)$$

Theorem 2.1.17 *The representation of $\mathbb{C}$ corresponding to $\hat{\mathbb{C}}$ is*

$$(\!(x_2,\{\emptyset,\pm\widehat{\infty}-y^\pm\})\!)=z(x_2,\{0,y^\pm\})\quad,$$

with Cartesian conversion functions

$$x(x_2,\{0,y^\pm\})=x_2\quad,$$

and

$$y(x_2,\{0,y^\pm\})=\begin{cases}\widehat{\infty}-y^+ & \textit{if}\quad \mathrm{Im}(z)>0\\ \quad 0 & \textit{if}\quad \mathrm{Im}(z)=0\\ y^- -\widehat{\infty} & \textit{if}\quad \mathrm{Im}(z)<0\end{cases}\quad.$$

Proof. All of $\mathbb{C}$ is in the domain of these functions. $\hat{\mathbb{C}}$ is piecewise defined (Definition 1.5.1) so it suffices to show that the pieces satisfy the definitions. The curly brackets in the $(\!(x_2,\{\emptyset,\pm\widehat{\infty}-y^\pm\})\!)$ notation refer to the three sets of variables in the piecewise definition of $\hat{\mathbb{C}}$ as

$$\begin{aligned}(\!(x_2,\emptyset)\!)&=z(x_2,0)\\ (\!(x_2,\widehat{\infty}-y^+)\!)&=z(x_2,y^+)\\ (\!(x_2,-\widehat{\infty}-y^-)\!)&=z(x_2,y^-)\quad.\end{aligned}$$

For $(\!(x,\emptyset)\!)$, we have conversion functions

$$x(x_2,0)=x_2\quad,\qquad\text{and}\qquad y(x_2,0)=0\quad.$$

Definition 2.1.12 gives

$$z_{(\!(x_2,\emptyset)\!)}[(\!(x,y)\!)]=z_{(\!(x_2,\emptyset)\!)}[x+iy]=x(x_2,0)+iy(x_2,0)=x_2\quad.$$

Therefore,

$$(\!(x_2,\emptyset)\!)=x_2\quad,\qquad\text{where}\qquad x_2\quad\equiv\quad x\quad.$$

For $(\!(x_2,\widehat{\infty}-y^+)\!)$, we have

$$f(x)=x\quad,\qquad\text{and}\qquad g(y)=\widehat{\infty}-y\quad.$$

with conversion functions

$$x(x_2,y^+)=x_2\quad,\qquad\text{and}\qquad y(x_2,y^+)=\widehat{\infty}-y^+\quad.$$

Definition 2.1.16 gives

$$\begin{aligned}z_{(\!(x_2,\widehat{\infty}-y^+)\!)}[(\!(x,y)\!)]&=z_{(\!(x_2,\widehat{\infty}-y^+)\!)}[x+iy]\\ &=z_{(\!(x_2,\widehat{\infty}-y^+)\!)}[f(x)+ig(y)]\\ &=z_{(\!(x_2,\widehat{\infty}-y^+)\!)}[x+i\big(\widehat{\infty}-y\big)]\end{aligned}$$

$$
\begin{aligned}
&= x(x_2, y^+) + i\big(\widehat{\infty} - y(x_2, y^+)\big) \\
&= x_2 + i\big[\widehat{\infty} - \big(\widehat{\infty} - y^+\big)\big] \quad .
\end{aligned}
$$

Since $y^+ \notin \mathbb{R}_0$, the quantity in parentheses is not an $\widehat{\mathbb{R}}$ number and the quantity in square brackets is not formatted for an additive composition of the form $\widehat{\infty} - \widehat{\mathbb{R}}$ given by Axiom 1.3.22. Substitute $y^+ = \widehat{\infty} - y$ (Definition 1.5.1) so that

$$
z_{(\!(x_2, \widehat{\infty} - y^+)\!)}[(\!(x, y)\!)] = x_2 + i\big\{\widehat{\infty} - \big[\widehat{\infty} - \big(\widehat{\infty} - y\big)\big]\big\} \quad .
$$

The quantity in square brackets obeys the additive composition laws for $\widehat{\infty} - \widehat{\mathbb{R}}$ (Axiom 1.3.22) so

$$
z_{(\!(x_2, \widehat{\infty} - y^+)\!)}[(\!(x, y)\!)] = x_2 + i\big(\widehat{\infty} - y\big) = x_2 + iy^+ \quad .
$$

Therefore,

$$
(\!(x_2, \widehat{\infty} - y^+)\!) = x_2 + iy^+ \quad .
$$

The final case is $(\!(x_2, -\widehat{\infty} - y^-)\!)$. We have

$$
f(x) = x \quad , \qquad \text{and} \qquad g(y) = -\widehat{\infty} - y \quad .
$$

with conversion functions are

$$
x(x_2, y^-) = x_2 \quad , \qquad \text{and} \qquad y(x_2, y^-) = y^- - \widehat{\infty} \quad .
$$

Definition 2.1.16 gives

$$
\begin{aligned}
z_{(\!(x_2, -\widehat{\infty} - y^-)\!)}[(\!(x, y)\!)] &= z_{(\!(x_2, -\widehat{\infty} - y^-)\!)}[x + iy] \\
&= z_{(\!(x_2, -\widehat{\infty} - y^-)\!)}[f(x) + ig(y)] \\
&= z_{(\!(x_2, -\widehat{\infty} - y^-)\!)}[x + i\big(-\widehat{\infty} - y\big)] \\
&= x(x_2, y^-) + i\big(-\widehat{\infty} - y(x_2, y^-)\big) \\
&= x_2 + i\big[-\widehat{\infty} - \big(y^- - \widehat{\infty}\big)\big] \quad .
\end{aligned}
$$

Since $y^- \notin \mathbb{R}$, the quantity in parentheses is not an $\widehat{\mathbb{R}}$ number and the quantity in square brackets is not formatted for an additive composition $\widehat{\infty} - \widehat{\mathbb{R}}$. Substitute $y^- = \widehat{\infty} + y$ (Definition 1.5.1) so that

$$
z_{(\!(x_2, -\widehat{\infty} - y^-)\!)}[(\!(x, y)\!)] = x_2 + i\big\{-\widehat{\infty} - \big[\big(\widehat{\infty} + y\big) - \widehat{\infty}\big]\big\} \quad .
$$

The quantity in square brackets obeys the additive composition laws for $\widehat{\mathbb{R}} + \widehat{\infty}$ so

$$
z_{(\!(x_2, -\widehat{\infty} - y^-)\!)}[(\!(x, y)\!)] = x_2 + i\big(-\widehat{\infty} - y\big) = x_2 - i\big(\widehat{\infty} + y\big) = x_2 - iy^- \quad .
$$

Therefore,

$$
(\!(x_2, -\widehat{\infty} - y^-)\!) = x_2 - iy^- \quad .
$$

We have proven that

$$\big(\!\big(x_2, \{\emptyset, \pm\widehat{\infty} - y^{\pm}\}\big)\!\big) = \begin{cases} x_2 + iy^{+} & \text{for} \quad \mathrm{Im}(z) > 0 \\ \quad x_2 & \text{for} \quad \mathrm{Im}(z) = 0 \\ x_2 - iy^{-} & \text{for} \quad \mathrm{Im}(z) < 0 \end{cases} \quad ,$$

and the theorem follows from $x \equiv x_2$.

Example 2.1.18 In this example, we show a case in which the representing functional correctly recovers the Cartesian representation from the $\hat{\mathbb{C}}$ representation. The conversion functions are

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^{+}(x,y) = \widehat{\infty} - y \quad .$$

The representing functional is

$$\begin{aligned} z_{(\!(x,y)\!)}\big[\big(\!\big(x, \widehat{\infty} - y^{+}\big)\!\big)\big] &= z_{(\!(x,y)\!)}\big[x + i\big(\widehat{\infty} - y^{+}\big)\big] \\ &= x(x,y) + i\big(\widehat{\infty} - y^{+}(x,y)\big) \\ &= x + i\big[\widehat{\infty} - \big(\widehat{\infty} - y\big)\big] \\ &= x + iy \quad . \end{aligned}$$

We have shown that the representing functional takes the $\hat{\mathbb{C}}$ representation and returns the Cartesian representation.

Remark 2.1.19 At this point, the reader hopefully is asking, "What is this convoluted notation for?" We introduce the rigorous representation to quantify what we mean by phrases like "the Cartesian representation of $\mathbb{C}$," "the polar representation of $\mathbb{C}$," or even "the $\hat{\mathbb{C}}$ representation of $\mathbb{C}$." For instance, we might wish to state precisely that the conversion functions of the Cartesian representation to the polar representation are analytic but the conversion functions of the Cartesian representation to the $\hat{\mathbb{C}}$ representation are one-to-one.

Example 2.1.20 As an illustration of the high significance of conversion functions, consider the Gaussian integral

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2} \quad .$$

This integral is analytically intractable in the Cartesian representation of $\mathbb{C}$ (except by quadrature) but it can be solved easily in the polar representation. We write canonically

$$I^2 = \int_{-\infty}^{\infty} dx\, e^{-x^2} \times \int_{-\infty}^{\infty} dx\, e^{-x^2} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-(x^2+y^2)} \quad ,$$

and then insert the conversion function

$$r(x,y) = \sqrt{x^2+y^2} \quad .$$

We obtain the infinitesimal element of polar area $dx \wedge dy = r\,dr \wedge d\theta$ from the conversion functions

$$x(r,\theta) = r\cos(\theta) \quad , \qquad \text{and} \qquad y(r,\theta) = r\sin(\theta) \quad ,$$

via

$$dx = \frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial \theta}\,d\theta \quad , \qquad \text{and} \qquad dy = \frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial \theta}\,d\theta \quad .$$

Then

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r e^{-r^2} \qquad \Longrightarrow \qquad I(z) = \sqrt{\pi} \quad .$$

§2.2 Definition of the representational derivative d/dz_1

Remark 2.2.1 To prove the limits of sine and cosine at infinity, we will use the definition of the derivative. First, we will compare the conventions for derivatives with respect to

$$z = x + iy \quad , \qquad \text{and} \qquad z = re^{i\theta} \quad ,$$

and then we will define derivatives with respect to the cases of

$$z = \begin{cases} x + iy^+ & \text{for} \quad \text{Im}(z) > 0 \\ \quad x & \text{for} \quad \text{Im}(z) = 0 \\ x - iy^- & \text{for} \quad \text{Im}(z) < 0 \end{cases} \quad .$$

We will use the definition of the representation to increase the specificity of the distinctions that we will make be specifically defining the derivative with respect to the variables of each representation. At the end of this paper, we will extract the limits of sine and cosine from the transformation of the derivatives among the representations.

Definition 2.2.2 The forward derivative of a complex-valued function is

$$\frac{d}{dz}\,f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad ,$$

where

$$\Delta z = z + \big(h - z\big) \quad , \qquad \text{and} \qquad h \in \mathbb{C} \quad .$$

Theorem 2.2.3 *The function $f(z) = e^z$ is an eigenfunction of the d/dz operator with unit eigenvalue.*

Proof. Using the definition of the derivative we find that

$$\begin{aligned}
\frac{d}{dz}\,e^z &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{z+\Delta z}-e^z}{\Delta z} \\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{x+iy+\Delta x+i\Delta y}-e^{x+iy}}{\Delta x+i\Delta y} \\
&= \lim_{\Delta y\to 0} \frac{e^{x+iy+i\Delta y}-e^{x+iy}}{i\Delta y} \\
&\overset{*}{=} \lim_{\Delta y\to 0} \frac{ie^{x+iy+i\Delta y}}{i} \\
&= e^z \quad .
\end{aligned}$$

(The $\overset{*}{=}$ symbol denotes an application of L'Hôpital's rule.)

Remark 2.2.4 The derivatives with respect to the polar and Cartesian representations are

$$\frac{d}{dz}f(z) = \lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0}} \frac{f(z+\Delta z)-f(z)}{\Delta z} \quad ,$$

and

$$\frac{d}{dz}f(z) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z+\Delta z)-f(z)}{\Delta z} \quad .$$

Δz is as in Definition 2.2.2:

$$\Delta z = z + \big(h-z\big) \quad , \qquad \text{and} \qquad h\in\mathbb{C} \quad .$$

There is usually no distinguishing between these two distinct instances of d/dz. We will do some tricks with these distinctions so it will be useful to distinguish the derivative with respect the each individual representation of complex numbers. For this reason, we will define the representational derivative d/dz_1.

Definition 2.2.5 The representational derivative

$$\frac{d}{dz_1}\,f(z_2) = \lim_{\substack{\Delta x_1\to 0\\ \Delta y_1\to 0}} \frac{f(z_2+\Delta z_2)-f(z_2)}{\Delta z_2} \quad ,$$

is such that the variables of the z_1 representation appear in the limit while the variables of z_2 appear in the limiting function.

Example 2.2.6 This example shows several cases of the representational derivative when $z_1 = z_2$. This example serves to specify the labels we will use for the different derivatives and functions. We have

$$\frac{d}{dz}f(z) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(z+\Delta z)-f(z)}{\Delta z} \quad , \qquad \text{for} \quad z(x,y)=x+iy$$

$$\frac{d}{dz'}f(z') = \lim_{\substack{\Delta r\to 0\\ \Delta \theta\to 0}}\frac{f(z'+\Delta z')-f(z')}{\Delta z'} \quad , \qquad \text{for} \quad z'(r,\theta)=re^{i\theta}$$

$$\frac{d}{dz^+}f(z^+) = \lim_{\substack{\Delta x\to 0\\ \Delta y^+\to 0}}\frac{f(z^++\Delta z^+)-f(z^+)}{\Delta z^+} \quad , \qquad \text{for} \quad z^+(x,y^+)=x+iy^+$$

$$\frac{d}{dz^-}f(z^-) = \lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}}\frac{f(z^-+\Delta z^-)-f(z^-)}{\Delta z^-} \quad , \qquad \text{for} \quad z^-(x,y^-)=x-iy^-$$

$$\frac{d}{dz^\emptyset}f(z^\emptyset) = \lim_{\Delta x\to 0}\frac{f(z^\emptyset+\Delta z^\emptyset)-f(z^\emptyset)}{\Delta z^\emptyset} \quad , \qquad \text{for} \quad z^\emptyset(x,\emptyset)=x \quad .$$

We will continue to use the labeling conventions on the right to describe the five main representations of $\mathbb{C}$: Cartesian, Polar, and the three pieces of the $\hat{\mathbb{C}}$ representation. The main purpose of the representational derivative regards the cases in which $z_1 \neq z_2$.

§2.3 Definition of the representational variation Δz_1

Definition 2.3.1 Following Definition 2.2.2, the variation of a $\mathbb{C}$ number in the definition of the representational derivative is

$$\Delta z_1 = z_1 + \big(h_1 - z_1\big) \quad , \quad h_1 \in \mathbb{C} \quad , \qquad \text{where} \qquad h \to 0 \quad .$$

The variation with respect to each representation has its own h_1 written in the analytical form of the z_1 representation. Δz_1 is called the representational variation.

Remark 2.3.2 The representational variation Δz_1 appears in each application of the representational derivative operator

$$\frac{d}{dz_1}f(z_1) = \lim_{\substack{\Delta x_1\to 0\\ \Delta y_1\to 0}}\frac{f(z_1+\Delta z_1)-f(z_1)}{\Delta z_1} \quad .$$

If we wish to compute a representational derivative of the form

$$\frac{d}{dz_1}f(z_2) = \lim_{\substack{\Delta x_1\to 0\\ \Delta y_1\to 0}}\frac{f(z_2+\Delta z_2)-f(z_2)}{\Delta z_2} \quad ,$$

then we will have to define Δz_2 in terms of the limiting variables x_1 and y_1.

Definition 2.3.3 The transformation law for the representational variation is

$$\Delta z_2(x_1, y_1) = \frac{\partial z_2}{\partial x_1}\,\Delta x_1 + \frac{\partial z_2}{\partial y_1}\,\Delta y_1 \quad .$$

With forethought for what we will call "the modified transformation law" in the next section, the present transformation will be called at times the canonical transformation law.

Definition 2.3.4 $\Delta z_2(x_1, y_1)$ is the variation of the z_2 representation of $\mathbb{C}$ written in terms of the variables of the z_1 representation. It appears in limits of the form

$$\frac{d}{dz_1}\,f(z_2) = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta y_1 \to 0}} \frac{f(z_2 + \Delta z_2(x_1, y_1)) - f(z_2)}{\Delta z_2} \quad .$$

Definition 2.3.5 The rules for computing the transformed variation are:

- Choose a transformation
$$\Delta z_2(x_1, y_1) = \frac{\partial z_2}{\partial x_1}\,\Delta x_1 + \frac{\partial z_2}{\partial y_1}\,\Delta y_1$$
- Write out the analytical form of z_2
$$\Delta z_2(x_1, y_1) = \frac{\partial}{\partial x_1}\big(z_2(x_2, y_2)\big)\Delta x_1 + \frac{\partial}{\partial y_1}\big(z_2(x_2, y_2)\big)\Delta y_1$$
- Replace x_2 and y_2 with their conversion functions
$$\begin{aligned}\Delta z_2(x_1, y_1) = &\frac{\partial}{\partial x_1}\big(z_2(x_2(x_1, y_1), y_2(x_1, y_1))\big)\Delta x_1 \\ &+ \frac{\partial}{\partial y_1}\big(z_2(x_2(x_1, y_1), y_2(x_1, y_1))\big)\Delta y_1\end{aligned}$$
- Simplify in terms of x_1 and y_1 to get the analytical form of
$$\Delta z_2(x_1, y_1)$$

Remark 2.3.6 Examples 2.3.7–2.3.10 and 2.3.12–2.3.13 demonstrate the transformed variation in several cases, several of which we will refer to later.

Example 2.3.7 This example shows the transformation of the Δz^+ variation into the variables of the Cartesian representation: $\Delta z^+(x, y)$. For the case of the derivative of a z^+ expression with respect to Cartesian z, we have

$$\frac{d}{dz}\,f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \Delta z^+) - f(z^+)}{\Delta z^+} \quad .$$

The analytical form of z^+ is

$$z^+(x, y^+) = x + iy^+ \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^+(x,y) = \widehat{\infty} - y \quad .$$

The transformation law for the variation is

$$\begin{aligned}
\Delta z^+(x,y) &= \frac{\partial z^+}{\partial x}\,\Delta x + \frac{\partial z^+}{\partial y}\,\Delta y \\
&= \frac{\partial}{\partial x}\big(x + iy^+\big)\Delta x + \frac{\partial}{\partial y}\big(x + iy^+\big)\Delta y \\
&= \frac{\partial}{\partial x}\big[x + i\big(\widehat{\infty} - y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x + i\big(\widehat{\infty} - y\big)\big]\Delta y \\
&= \Delta x - i\Delta y \quad .
\end{aligned}$$

Example 2.3.8 This example shows the transformation of the Δz^- variation into the variables of the Cartesian representation: $\Delta z^-(x,y)$. For the case of the derivative of a z^- expression with respect to Cartesian z, we have

$$\frac{d}{dz}\,f(z^-) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^- + \Delta z^-) - f(z^-)}{\Delta z^-} \quad .$$

The analytical form of z^- is

$$z^-(x, y^-) = x - iy^- \quad ,$$

with two conversion functions

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^-(x,y) = \widehat{\infty} + y \quad .$$

The transformation law for the variation is

$$\begin{aligned}
\Delta z^-(x,y) &= \frac{\partial z^-}{\partial x}\,\Delta x + \frac{\partial z^-}{\partial y}\,\Delta y \\
&= \frac{\partial}{\partial x}\big(x - iy^-\big)\Delta x + \frac{\partial}{\partial y}\big(x - iy^-\big)\Delta y \\
&= \frac{\partial}{\partial x}\big[x - i\big(\widehat{\infty} + y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x - i\big(\widehat{\infty} + y\big)\big]\Delta y \\
&= \Delta x - i\Delta y \quad .
\end{aligned}$$

Example 2.3.9 This example shows the transformation of the Cartesian Δz variation into the variables of the z^+ representation: $\Delta z(x, y^+)$. For the case of the derivative of a z expression with respect to z^+, we have

$$\frac{d}{dz^+}\,f(z) = \lim_{\substack{\Delta x\to 0\\ \Delta y^+\to 0}} \frac{f(z+\Delta z)-f(z)}{\Delta z} \quad .$$

The analytical form of z is

$$z(x,y) = x+iy \quad ,$$

with two conversion functions

$$x(x,y^+) = x \quad , \qquad \text{and} \qquad y(x,y^+) = \widehat{\infty} - y^+ \quad .$$

The transformation law for the variation is

$$\begin{aligned}
\Delta z(x,y^+) &= \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y^+}\,\Delta y^+ \\
&= \frac{\partial}{\partial x}\big(x+iy\big)\Delta x + \frac{\partial}{\partial y^+}\big(x+iy\big)\Delta y^+ \\
&= \frac{\partial}{\partial x}\big[x+i\big(\widehat{\infty}-y^+\big)\big]\Delta x + \frac{\partial}{\partial y^+}\big[x+i\big(\widehat{\infty}-y^+\big)\big]\Delta y^+ \\
&= \Delta x - i\Delta y^+ \quad .
\end{aligned}$$

Example 2.3.10 This example shows the transformation of the Cartesian Δz variation into the variables of the z^- representation: $\Delta z(x, y^-)$. For the case of the derivative of a z expression with respect to z^-, we have

$$\frac{d}{dz^-}\,f(z) = \lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}} \frac{f(z+\Delta z)-f(z)}{\Delta z} \quad .$$

The analytical form of z is

$$z(x,y) = x+iy \quad ,$$

with two conversion functions

$$x(x,y^-) = x \quad , \qquad \text{and} \qquad y(x,y^-) = y^- - \widehat{\infty} \quad .$$

The transformation law for the variation is

$$\begin{aligned}
\Delta z(x,y^-) &= \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y^-}\,\Delta y^- \\
&= \frac{\partial}{\partial x}\big(x+iy\big)\Delta x + \frac{\partial}{\partial y^-}\big(x+iy\big)\Delta y^- \\
&= \frac{\partial}{\partial x}\big[x+i\big(y^- -\widehat{\infty}\big)\big]\Delta x + \frac{\partial}{\partial y^-}\big[x+i\big(y^- -\widehat{\infty}\big)\big]\Delta y^- \\
&= \Delta x + i\Delta y^- \quad .
\end{aligned}$$

Remark 2.3.11 Notice that the variation is the same between the two cases of z^+ (Examples 2.3.7 and 2.3.9):

$$\Delta z^+(x,y) = \Delta x - i\Delta y \quad , \qquad \text{and} \qquad \Delta z(x,y^+) = \Delta x - i\Delta y^+ \quad ,$$

but, to the contrary, the sign changes between the conversions to and from z^- (Examples 2.3.8 and 2.3.10):

$$\Delta z^-(x,y) = \Delta x - i\Delta y^- \quad , \qquad \text{and} \qquad \Delta z(x,y^-) = \Delta x + i\Delta y^- \quad .$$

Example 2.3.12 This example shows the transformation of the polar $\Delta z'$ variation into the variables of the Cartesian z representation: $\Delta z'(x,y)$. For the case of

$$\frac{d}{dz} f(z') = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z'+\Delta z') - f(z')}{\Delta z'} \quad ,$$

we have

$$z'(r,\theta) = re^{i\theta} \quad ,$$

with two conversion functions

$$r(x,y) = \sqrt{x^2+y^2} \quad , \qquad \text{and} \qquad \theta(x,y) = \tan^{-1}\left(\frac{y}{x}\right) \quad .$$

The transformation law for the variation is

$$\begin{aligned} \Delta z'(x,y) &= \frac{\partial z'}{\partial x}\Delta x + \frac{\partial z'}{\partial y}\Delta y \\ &= \frac{\partial}{\partial x}\big(re^{i\theta}\big)\Delta x + \frac{\partial}{\partial y}\big(re^{i\theta}\big)\Delta y \end{aligned}$$

We have shown in Example 2.1.14 that the conversion functions yield $x+iy$ so

$$\begin{aligned} \Delta z'(x,y) &= \frac{\partial}{\partial x}\big(x+iy\big)\Delta x + \frac{\partial}{\partial y}\big(x+iy\big)\Delta y \\ &= \Delta x + i\Delta y \quad . \end{aligned}$$

Example 2.3.13 This example shows the transformation of the Cartesian Δz variation into the variables of polar z' representation: $\Delta z(r,\theta)$. For the case of

$$\frac{d}{dz'} f(z) = \lim_{\substack{\Delta r \to 0 \\ \Delta \theta \to 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad ,$$

we have

$$z(x,y) = x + iy \quad ,$$

with two conversion functions

$$x(r,\theta) = r\cos(\theta) \quad , \qquad \text{and} \qquad y(r,\theta) = r\sin(\theta) \quad .$$

The transformation law for the variation is

$$\begin{aligned}\Delta z(r,\theta) &= \frac{\partial z}{\partial r}\,\Delta r + \frac{\partial z}{\partial \theta}\,\Delta\theta \\ &= \frac{\partial}{\partial r}\big(x+iy\big)\Delta r + \frac{\partial}{\partial \theta}\big(x+iy\big)\Delta\theta \quad .\end{aligned}$$

We have shown in Example 2.1.13 that the conversion functions yield $x+iy = re^{i\theta}$ so

$$\begin{aligned}\Delta z(r,\theta) &= \frac{\partial}{\partial r}\big(re^{i\theta}\big)\Delta r + \frac{\partial}{\partial \theta}\big(re^{i\theta}\big)\Delta\theta \\ &= e^{i\theta}\Delta r + ire^{i\theta}\Delta\theta \quad .\end{aligned}$$

Remark 2.3.14 Examples 2.3.15–2.3.18 all consider the same function in four different representational derivatives and then, for breadth, we will demonstrate the derivative of the logarithm in Example 2.3.19. A case of the chain rule appears in Example 2.3.20.

Example 2.3.15 Consider the function $f(z) = 3z^2$ and its representational derivative

$$\frac{d}{dz}\,f(z^+) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z^+ + \Delta z^+) - f(z^+)}{\Delta z^+} \quad .$$

The conversion functions are

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^+(x,y) = \widehat{\infty} - y \quad .$$

The transformation law for the variation is (Example 2.3.7)

$$\Delta z^+(x,y) = \Delta x - i\Delta y \quad .$$

Evaluation yields

$$\begin{aligned}\frac{d}{dz}\,3\big(z^+\big)^2 &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{3\big(z^+ + \Delta z^+\big)^2 - 3\big(z^+\big)^2}{\Delta z^+} \\ &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{3\big(x + iy^+ + \Delta z^+\big)^2 - 3\big(x+iy^+\big)^2}{\Delta z^+} \\ &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{3\big[x + i\big(\widehat{\infty} - y\big) + \Delta x - i\Delta y\big]^2 - 3\big[x - i\big(\widehat{\infty} - y\big)\big]^2}{\Delta x - i\Delta y}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\Delta y \to 0} \frac{3\big[x + i\big(\widehat{\infty} - y\big) - i\Delta y\big]^2 - 3\big[x + i\big(\widehat{\infty} - y\big)\big]^2}{-i\Delta y} \\
&\stackrel{*}{=} \lim_{\Delta y \to 0} \frac{-6i\big[x + i\big(\widehat{\infty} - y\big) - i\Delta y\big]}{-i} \\
&= 6\big[x + i\big(\widehat{\infty} - y\big)\big] = 6\big(x + iy^+\big) = 6z^+ \quad .
\end{aligned}
$$

This example has demonstrated the validity of the transformation law for the variation.

Example 2.3.16 Consider the function $f(z) = 3z^2$ and its representational derivative

$$
\frac{d}{dz'} f(z) = \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad .
$$

The conversion functions are

$$
x(r,\theta) = r\cos(\theta) \quad , \qquad \text{and} \qquad y(r,\theta) = r\sin(\theta) \quad .
$$

The transformation law for the variation is (Example 2.3.13)

$$
\Delta z(r,\theta) = e^{i\theta}\Delta r + ire^{i\theta}\Delta\theta
$$

Converting to polar gives

$$
\begin{aligned}
\frac{d}{dz'} 3z^2 &= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \frac{3\big(z(r,\theta) + \Delta z(r,\theta)\big)^2 - 3\big(z(r,\theta)\big)^2}{\Delta z'} \\
&= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0}} \frac{3\big(re^{i\theta} + e^{i\theta}\,\Delta r + ire^{i\theta}\,\Delta\theta\big)^2 - 3\big(re^{i\theta}\big)^2}{e^{i\theta}\,\Delta r + ire^{i\theta}\,\Delta\theta} \\
&= \lim_{\Delta\theta \to 0} \frac{3\big(re^{i\theta} + ire^{i\theta}\,\Delta\theta\big)^2 - 3\big(re^{i\theta}\big)^2}{ire^{i\theta}\,\Delta\theta} \\
&\stackrel{*}{=} \lim_{\Delta\theta \to 0} \frac{6ire^{i\theta}\big(re^{i\theta} + ire^{i\theta}\,\Delta\theta\big)}{ire^{i\theta}} \\
&= 6\big(re^{i\theta}\big) = 6z \quad .
\end{aligned}
$$

We have the correct transformation law for Δz.

Example 2.3.17 Consider the function $f(z) = 3z^2$ and its representational derivative

$$
\frac{d}{dz} f(z^\emptyset) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^\emptyset + \Delta z^\emptyset) - f(z^\emptyset)}{\Delta z^\emptyset} \quad .
$$

The conversion functions are

$$x(x,\emptyset) = x \quad , \qquad \text{and} \qquad y^\emptyset(x,\emptyset) = 0 \quad .$$

The transformation law for the variation is

$$\Delta z^\emptyset(x,y) = \frac{\partial}{\partial x}\big(x\big)\Delta x = \Delta x \quad .$$

Evaluation yields

$$\frac{d}{dz}\,3\big(z^\emptyset\big)^2 = \lim_{\Delta x\to 0}\frac{3\big(x+\Delta x\big)^2 - 3\big(x\big)^2}{\Delta x} \stackrel{*}{=} \lim_{\Delta x\to 0} 6\big(x+\Delta x\big) = 6x = 6z^\emptyset \quad .$$

Example 2.3.18 Consider the function $f(z) = 3z^2$ and its representational derivative

$$\frac{d}{dz^-}\,f(z) = \lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}}\frac{f(z+\Delta z)-f(z)}{\Delta z} \quad ,$$

The conversion functions are

$$x(x,y^-) = x \quad , \qquad \text{and} \qquad y(x,y^-) = y^- - \widehat{\infty} \quad .$$

The transformation law for the variation is (Example 2.3.10)

$$\Delta z(x,y^-) = \Delta x + i\Delta y^- \quad .$$

Evaluation yields

$$\begin{aligned}
\frac{d}{dz^-}\,3\big(z\big)^2 &= \lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}}\frac{3\big(z+\Delta z\big)^2 - 3\big(z\big)^2}{\Delta z}\\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}}\frac{3\big(x+iy+\Delta z\big)^2 - 3\big(x+iy\big)^2}{\Delta z}\\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y^-\to 0}}\frac{3\big[x+i\big(y^- - \widehat{\infty}\big)+\Delta x + i\Delta y\big]^2 - 3\big[x-i\big(y^- - \widehat{\infty}\big)\big]^2}{\Delta x + i\Delta y}\\
&= \lim_{\Delta y^-\to 0}\frac{3\big[x+i\big(y^- - \widehat{\infty}\big)+ i\Delta y\big]^2 - 3\big[x+i\big(y^- - \widehat{\infty}\big)\big]^2}{i\Delta y}\\
&\stackrel{*}{=} \lim_{\Delta y^-\to 0}\frac{6i\big[x+i\big(y^- - \widehat{\infty}\big)+ i\Delta y\big]}{i}\\
&= 6\big[x+i\big(y^- - \widehat{\infty}\big)\big] = 6\big(x+iy\big) = 6z \quad .
\end{aligned}$$

This example has demonstrated the validity of the transformation law for the variation.

Example 2.3.19 Consider the function $f(z) = \ln(z)$ and its representational derivative

$$\frac{d}{dz} f(z^-) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^- + \Delta z^-) - f(z^-)}{\Delta z^-} \quad .$$

The conversion functions are

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^-(x,y) = \widehat{\infty} + y \quad .$$

The transformation law for the variation is (Example 2.3.8)

$$\Delta z^-(x,y) = \Delta x - i\Delta y \quad .$$

Evaluation yields

$$\begin{aligned}
\frac{d}{dz} \ln(z^-) &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\ln(z^- + \Delta z^-) - \ln(z^-)}{\Delta z^-} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\ln(x - iy^- + \Delta x - i\Delta y) - \ln(x - iy^-)}{\Delta x - i\Delta y} \\
&= \lim_{\Delta x \to 0} \frac{\ln(x - iy^- + \Delta x) - \ln(x - iy^-)}{\Delta x} \\
&\stackrel{*}{=} \lim_{\Delta x \to 0} \frac{1}{x - iy^- + \Delta x} \\
&= \frac{1}{x - iy^-} = \frac{1}{z^-} \quad .
\end{aligned}$$

Example 2.3.20 Consider the derivative of $f(z) = 3ze^{2z}$ in the form

$$\frac{d}{dz^+} f(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad .$$

The conversion functions are

$$x(x,y^+) = x \quad , \qquad \text{and} \qquad y(x,y^+) = \widehat{\infty} - y^+ \quad .$$

The transformation law for the variation is (Example 2.3.9)

$$\Delta z(x,y^+) = \Delta x - i\Delta y^+ \quad .$$

Evaluation yields

$$\frac{d}{dz^+} 3ze^{2z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y^+ \to 0}} \frac{3\big(z + \Delta z\big)\, e^{2(z + \Delta z)} - 3ze^{2z}}{\Delta z}$$

$$
\begin{aligned}
&= \lim_{\substack{\Delta x\to 0\\ \Delta y^+\to 0}} \frac{3\big(x+iy+\Delta z\big)\,e^{2(x+iy+\Delta z)}-3\big(x+iy\big)e^{2(x+iy)}}{\Delta z}\\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y^+\to 0}} \left\{\frac{3\big[x+i\big(\widehat{\infty}-y^+\big)+\Delta x-i\Delta y^+\big]\,e^{2[x+i(\widehat{\infty}-y^+)+\Delta x-i\Delta y^+]}}{\Delta x-i\Delta y^+}\right.\\
&\qquad\left. -\frac{3\big[x+i\big(\widehat{\infty}-y^+\big)\big]\,e^{2[x+i(\widehat{\infty}-y^+)]}}{\Delta x-i\Delta y^+}\right\}\\
&= e^{2[x+i(\widehat{\infty}-y^+)]}\lim_{\Delta y^+\to 0}\left\{\frac{3\big[x+i\big(\widehat{\infty}-y^+\big)-i\Delta y^+\big]\,e^{-2i\Delta y^+}}{-i\Delta y^+}\right.\\
&\qquad\left. -\frac{3\big[x+i\big(\widehat{\infty}-y^+\big)\big]}{-i\Delta y^+}\right\}\\
&\overset{*}{=} e^{2[x+i(\widehat{\infty}-y^+)]}\lim_{\Delta y^+\to 0}\left\{\frac{-3ie^{-2i\Delta y^+}}{-i}\right.\\
&\qquad\left. -\frac{6i\big[x+i\big(\widehat{\infty}-y^+\big)-i\Delta y^+\big]\,e^{-2i\Delta y^+}}{-i}\right\}\\
&= e^{2[x+i(\widehat{\infty}-y^+)]}\Big\{3+6\big[x+i\big(\widehat{\infty}-y^+\big)\big]\Big\}\\
&= e^{2[x+i(\widehat{\infty}-y^+)]}\Big[3+6\big(x+iy\big)\Big]=e^{2z}\big(3+6z\big) \quad .
\end{aligned}
$$

Theorem 2.3.21 *The complex exponential function e^z is an eigenfunction of the representational derivative operator d/dz_1.*

Proof. It suffices to show that

$$
\frac{d}{dz_1}\,e^{z_1}=e^{z_1} \quad , \qquad \text{and} \qquad \frac{d}{dz_1}\,e^{z_2}=e^{z_2} \quad ,
$$

where z_1 and z_2 are two different representations of $\mathbb{C}$. The first condition is satisfied by Theorem 2.2.3. For the second condition, consider

$$
\frac{d}{dz}\,f(z^+)=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(z^++\Delta z^+)-f(z^+)}{\Delta z^+} \quad ,
$$

with two conversion functions

$$
x(x,y)=x \quad , \qquad \text{and} \qquad y^+(x,y)=\widehat{\infty}-y \quad .
$$

Example 2.3.7 gives the transformation law for the variation as

$$
\Delta z^+(x,y)=\Delta x-i\Delta y \quad .
$$

Evaluation yields

$$\begin{aligned}
\frac{d}{dz}\,e^{z^+} &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{z^+ + \Delta z^+} - e^{z^+}}{\Delta z^+} \\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{x+iy^+ + \Delta z^+} - e^{x+iy^+}}{\Delta z^+} \\
&= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{x+i(\widehat{\infty}-y)+\Delta x - i\Delta y} - e^{x+i(\widehat{\infty}-y)}}{\Delta x - i\Delta y} \\
&= \lim_{\Delta x\to 0} \frac{e^{x+i(\widehat{\infty}-y)+\Delta x} - e^{x+i(\widehat{\infty}-y)}}{\Delta x} \\
&\stackrel{*}{=} \lim_{\Delta x\to 0} e^{x+i(\widehat{\infty}-y)+\Delta x} \\
&= e^{x+i(\widehat{\infty}-y)} = e^{x+iy^+} = e^{z^+} \quad .
\end{aligned}$$

All other cases of z_1, z_2 follow directly.

Remark 2.3.22 We have shown that the transformation law for the variation with respect to each representation produces the correct derivative. Since the formula

$$\Delta z_2(x_1, y_1) = \frac{\partial z_2}{\partial x_1}\,\Delta x_1 + \frac{\partial z_2}{\partial y_1}\,\Delta y_1 \quad ,$$

given in Definition 2.3.3 is totally standard, this is as expected. In the following section, we will examine the definition of the variation and propose a modified variation which obeys a separate transformation law. We will show that the two transformations do not always agree and that the transformation of the modified variation does not always work for the definition of the derivative; sometimes using the modified transformation will break the formula. Then we will show that the transformation of the modified variation *does* always produce the correct derivative when the transformation is from the $z^\pm$ representations to the Cartesian representation. This will be due to the composition laws of $\widehat{\mathbb{R}}$ and the properties of $\widehat{\infty}$.

§2.4 Definition of the modified representational variation $\widehat{\Delta} z_1$

Remark 2.4.1 In Definition 2.2.2, we stated the definition of the derivative d/dz which includes by construction the variation Δz. In Definition 2.2.5, we introduced the representational derivative d/dz_1 which makes specific distinctions about which representations of $\mathbb{C}$ we will use when taking a specific derivative. The representational derivative required the introduction of the representational variation Δz_1. In Definition 2.3.3, we gave the transformation law required for Δz_1 to accommodate the structure of the representational derivative There we gave nothing new; we simply stated the ordinary transformation law which is well known. In this section, we will propose for the variation another transformation law based on geometric rather than algebraic considerations. Then we will prove for a certain case that this modified transformation always gives the correct derivative for representational derivative. The notation will be such that $\widehat{\Delta z}_1$ has the same $z + (h - z)$ structure as Δz_1 but the hat tells us to use the modified transformation law defined in this section.

Definition 2.4.2 The modified representational variation of a $\mathbb{C}$ number is

$$\widehat{\Delta z}_1 = z_1 + \big(h_1 - z_1\big) \quad , \quad h \in \mathbb{C} \quad , \quad h \to 0 \quad ,$$

so it is identically the representational variation Δz_1 (Definition 2.3.1.) The difference between Δz_1 and $\widehat{\Delta z}_1$ is that they obey different transformation laws among representations.

Definition 2.4.3 The modified representational derivative is

$$\widehat{\frac{d}{dz_1}} f(z_2) = \lim_{\Delta z_1 \to 0} \frac{f(z_2 + \widehat{\Delta z}_2) - f(z)}{\widehat{\Delta z}_2} \quad ,$$

The hat on the derivative operator tells us to use the modified variation.

Axiom 2.4.4 The modified representational variation $\widehat{\Delta z}_1$ transforms by inserting the conversion functions directly into the definition of the variation.

Definition 2.4.5 The rules for computing the transformed modified variation are:

- Choose a modified transformation

$$\widehat{\Delta z}_2(x_1, y_1) = z_2 + \big(h_2 - z_2\big)$$

- Write out the analytical form of z_2 and h_2

$$\widehat{\Delta z}_2(x_1, y_1) = z_2(x_2, y_2) + \big(h_2(x_2, y_2) - z_2(x_2, y_2)\big)$$

(The relevant analytical forms are:

$$\begin{aligned}
z_2 &= z && \Longrightarrow && z_2 = x + iy \\
z_2 &= z' && \Longrightarrow && z_2 = re^{i\theta} \\
z_2 &= z^{\emptyset} && \Longrightarrow && z_2 = x \\
z_2 &= z^{+} && \Longrightarrow && z_2 = x + iy^{+} \\
z_2 &= z^{-} && \Longrightarrow && z_2 = x - iy^{-} \quad .)
\end{aligned}$$

- Replace x_2 and y_2 with their conversion functions

$$\widehat{\Delta z}_2(x_1,y_1) = z_2(x_2(x_1,y_1),y_2(x_1,y_1)) \\ \qquad + \big(h_2(x_2(a,b),y_2(a,b)) - z_2(x_2(x_1,y_1),y_2(x_1,y_1))\big)$$

- Simplify in terms of x_1 and y_1 to get the analytical form of

$$\widehat{\Delta z}_2(x_1,y_1)$$

Example 2.4.6 The example gives the geometric motivation for the modified transformation law obeyed by the modified representational variation $\widehat{\Delta z_1}$. To begin, we will consider the algebraic structure of the variation which appears in the definition of the derivative. While we have so far examined the 2D case of the variation on the complex plane, the 1D case of

$$\Delta x = x + \big(h - x\big) \quad , \quad h \in \mathbb{R}_0 \quad , \quad h \to 0 \quad .$$

gives the best illustration of the meaning of the definition of the variation. Considering three sequential points $\{\mathcal{O}, x, h\}$ along the real number line such that $\mathcal{O}$ is the origin and

$$0 < x < h \quad .$$

We could shift the origin to any other $y \in \mathbb{R}$ and then write the definition of the variation with respect to those coordinates using three sequential points $\{\mathcal{O}', x', h'\}$ such that

$$0' < x' < h' \quad .$$

For instance, if we shift $h \to h' = y + h$, then $h \to 0$ no longer generates an appropriate variation because

$$\lim_{h\to 0} h' = \lim_{h\to 0} y + h = y \quad ,$$

is not vanishingly small when y is not vanishingly small. To get the correct derivative for arbitrary y, we need to take $h' \to 0$. This requires that h goes to $-y$. By the symmetry of the real line, either of these representations of the 1D variation Δx, that built around the origin $\mathcal{O}$ and that built around the translated origin $\mathcal{O}' = \mathcal{O} + y$, are exactly the same. Despite this identical sameness, if we use a translation operator to directly transform Δz_1 without following the prescription given by Definition 2.3.3, the derivative does not always work out correctly. However, we will prove that we *do* always get the correct derivative in at least one certain case.

Remark 2.4.7 Example 2.4.8 example shows that the canonical transformation law of Definition 2.3.3 and the modified transformation law of Axiom 2.4.4 produce two unequal expressions. After this example, we will proves a case in which the derivative remains invariant between d/dz_1 and $\widehat{d}/dz_1$.

Example 2.4.8 This example shows that the canonical transformation law of Definition 2.3.3 and the modified transformation law of Axiom 2.4.4 produce two unequal expressions. To that end, define a representation of $\mathbb{C}$ such that

$$z_{(\!(x,y^\gamma)\!)}[(\!(x,y)\!)] = (\!(x,y^\gamma)\!) = x + iy^\gamma \quad .$$

Suppose the two conversion functions are

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^\gamma(x,y) = \gamma - y \quad , \quad \gamma \in \mathbb{R}_0 \quad ,$$

(The z^γ representation is like the z^+ representation with the origin of the imaginary axis shifted by a finite amount γ rather than the infinite amount seen in $y^+ = \widehat{\infty} - y$.) It is required for this example to demonstrate the transformation of the variation by direct application of the conversion functions to the elements of

$$\widehat{\Delta z}_1 = z_1 + \big(h_1 - z_1\big) \quad .$$

We will do so in the case of

$$\frac{\widehat{d}}{dz} f(z^\gamma) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^\gamma + \widehat{\Delta z}^\gamma) - f(z^\gamma)}{\widehat{\Delta z}^\gamma} \quad .$$

This expression requires that we transform the modified variation from the z^γ representation to the Cartesian representation. To do so, we must write recognize that h^γ has the analytical form of the form of the z^γ representation

$$z^\gamma = x + iy^\gamma \qquad \Longrightarrow \qquad h^\gamma = a^\gamma + ib^\gamma \quad .$$

According to Axiom 2.4.4, the modified variation transforms by inserting the conversion functions directly into the definition of the variation:

$$\begin{aligned}
\widehat{\Delta z}^\gamma(x,y) &= z^\gamma + \big(h^\gamma - z^\gamma\big) \\
&= z^\gamma(x,y^\gamma) + \big(h^\gamma - z^\gamma(x,y^\gamma)\big) \\
&= x(x,y) + iy^\gamma(x,y) + \big[a^\gamma + ib^\gamma - \big(x(x,y) + iy^\gamma(x,y)\big)\big] \\
&= x + i\big(\gamma - y\big) + \big\{a + i\big(\gamma - b\big) - \big[x + i\big(\gamma - y\big)\big]\big\} \\
&= x + i\gamma - iy + a + i\gamma - ib - x - i\gamma + iy \\
&= \big(x + a - x\big) + \big(iy - ib + iy\big) + i\gamma \\
&= \Delta x - i\Delta y + i\gamma \quad .
\end{aligned}$$

The transformation law for the canonical variation Δz (Definition 2.3.3) gives

$$\begin{aligned}
\Delta z^\gamma(x,y) &= \frac{\partial}{\partial x}\big[x + i\big(\gamma - y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x + i\big(\gamma - y\big)\big]\Delta y \\
&= \Delta x - i\Delta y \quad .
\end{aligned}$$

We find

$$\widehat{\Delta z}^{\gamma}(x,y) = \Delta z^{\gamma}(x,y) + i\gamma \quad .$$

The two transformation laws do not produce the same transformed values because

$$\widehat{\Delta z}^{\gamma}(x,y) \neq \Delta z^{\gamma}(x,y) \quad .$$

Remark 2.4.9 The transformation law for the variation does not agree with our attempt to transform the modified variation by directly converting its elements with the conversion functions. We will show, however, that this not a problem in all cases. Example 2.4.10 shows that the modified variation breaks the definition of the derivative for finite γ and the remainder of this section builds a proof that the modified variation does not always break the derivative when $\gamma \to \widehat{\infty}$.

Example 2.4.10 Consider the function $f(z) = z^3$ and the representational derivative

$$\frac{\widehat{d}}{dz} f(z^{\gamma}) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^{\gamma} + \widehat{\Delta z}^{\gamma}) - f(z^{\gamma})}{\widehat{\Delta z}^{\gamma}} \quad .$$

In Example 2.4.8, we derived the expression

$$\widehat{\Delta z}^{\gamma}(x,y) = \Delta x - i\Delta y + i\gamma \quad .$$

Let $\gamma = 2$ so the derivative becomes

$$\begin{aligned}
\frac{\widehat{d}}{dz} \big(z^{\gamma}\big)^3 &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\big(z^{\gamma} + \widehat{\Delta z}^{\gamma}\big)^3 - \big(z^{\gamma}\big)^3}{\widehat{\Delta z}^{\gamma}} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\big(x + iy^{\gamma} + \Delta x - i\Delta y + 2i\big)^3 - \big(x + iy^{\gamma}\big)^3}{\Delta x - i\Delta y + 2i} \\
&= \frac{\big[x + i\big(\widehat{\infty} - y\big) + 2i\big]^3 - \big[x + i\big(\widehat{\infty} - y\big)\big]^3}{2i} \\
&\neq 3\big[x + \big(\widehat{\infty} - y\big)\big]^2 = 3\big(z^{\gamma}\big)^2 \quad .
\end{aligned}$$

Remark 2.4.11 Example 2.4.10 proves that the modified variation breaks the definition of the derivative for at least some finite γ. We cannot prove by example that the derivative always works for $\gamma = \widehat{\infty}$ so we will construct a proof for a specific case in the remainder of this section. We want the proof to hold for z^+ and z^- so we need to compute the modified variation for the z^- part of the $\hat{\mathbb{C}}$ representation. (The z^+ version can be extracted directly from

Example 2.4.8.) We will do so in Example 2.4.12 and then we will compute the inverse transformations in Example 2.4.13.

Example 2.4.12 To transform $\widehat{\Delta z}^-$ as required for a Cartesian derivative $\hat{d}/dz_1$, we have two conversion functions

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^-(x,y) = \widehat{\infty} + y \quad ,$$

The modified transformation law (Definition 2.4.5) is

$$\begin{aligned}\widehat{\Delta z}^-(x,y) &= z^- + \big(h^- - z^-\big) \\ &= z^-(x,y^-) + \big(h^-(x,y^-) - z^-(x,y^-)\big) \\ &= z^-(x(x,y),y^-(x,y)) \\ &\qquad + \big(h^-(x(a,b),y^-(a,b)) - z^-(x(x,y),y^-(x,y))\big) \quad .\end{aligned}$$

Since we seek to replicate the z^- representation, we must give h^- te analytical form of that representation: $h^- = a^- - ib^-$. It follows that

$$\begin{aligned}\widehat{\Delta z}^-(x,y) &= x(x,y) - iy^-(x,y) + \big[x(a,b) - iy^-(a,b) - \big(x(x,y) - iy^-(x,y)\big)\big] \\ &= x - i\big(\widehat{\infty} + y\big) + \big\{a - i\big(\widehat{\infty} + b\big) - \big[x - i\big(\widehat{\infty} + y\big)\big]\big\} \quad .\end{aligned}$$

If b is positive, then by Definition 1.3.8 we have obtained an undefined expression $\widehat{\infty} + b$. We ensure that b is negative through the generalization of the three points $\{\mathcal{O}, x, h\}$ given in Example 2.4.8. We have given them an ordering $0 < x < h$ based on the supposed positivity of x. When $y > 0$, we have three points $\{\mathcal{O}, y, b\}$ which are ordered such that $b < y < 0$. This guarantees that $\widehat{\infty} + b$ is a defined quantity. It is an element of $\widehat{\mathbb{R}}$ (Definition 1.3.3.) Continuing the calculation yields

$$\begin{aligned}\widehat{\Delta z}^-(x,y) &= x - i\big(\widehat{\infty} + y\big) + \big\{a - i\big(\widehat{\infty} + b\big) - \big[x - i\big(\widehat{\infty} + y\big)\big]\big\} \\ &= x - i\big(\widehat{\infty} + y\big) + \big[a - i\big(\widehat{\infty} + b\big) - x + i\big(\widehat{\infty} + y\big)\big] \\ &= x - i\big(\widehat{\infty} + y\big) + \big\{a - x - i\big[\big(\widehat{\infty} + b\big) - \big(\widehat{\infty} + y\big)\big]\big\} \quad .\end{aligned}$$

At this point, we may not follow exactly the algebraic steps used to derive $\widehat{\Delta z}^\gamma(x,y)$ in Example 2.4.8 because we have to use Axiom 1.3.19 to compute the difference of two $\widehat{\mathbb{R}}$ numbers. For Example 2.4.8, however, it is obvious that the expression for $\widehat{\Delta z}^\gamma(x,y)$ correctly generalizes to $\widehat{\Delta z}^+(x,y)$ because the answer comes out the same if we follow the order of operations given by the bracketing. (Per Axiom 1.3.19, this order must be followed because $\widehat{\mathbb{R}} + \widehat{\mathbb{R}}$ is not an associative operation.) Applying Axiom 1.3.19 to the present derivation yields

$$\widehat{\Delta z}^-(x,y) = x - i\big(\widehat{\infty} + y\big) + \big[a - x - i\big(b - y\big)\big]$$

$$
\begin{aligned}
&= x - i\widehat{\infty} - iy + a - x - ib + iy \\
&= \big(x + a - x\big) - i\big(y + b - y\big) - i\widehat{\infty} \\
&= \Delta x - i\Delta y - i\widehat{\infty}
\end{aligned}
$$

This does not agree with the canonical transformation of the variation (Definition 2.3.3)

$$
\begin{aligned}
\Delta z^{-}(x,y) &= \frac{\partial}{\partial x}\big[x - i\big(\widehat{\infty} + y\big)\big]\Delta x + \frac{\partial}{\partial y}\big[x - i\big(\widehat{\infty} + y\big)\big]\Delta y \\
&= \Delta x - i\Delta y \quad .
\end{aligned}
$$

Example 2.4.13 Examples 2.4.8 and 2.4.12 gave the transformation of $\widehat{\Delta z}^{\pm}$ into the Cartesian representation. In this example, we will transform $\widehat{\Delta z}$ into the $z^{\pm}$ components of the $\hat{\mathbb{C}}$ representation. For $\widehat{\Delta z}(x,y^{+})$, we have two conversion functions

$$
x(x,y^{+}) = x \quad , \qquad \text{and} \qquad y(x,y^{+}) = \widehat{\infty} - y^{+} \quad .
$$

Definition 2.4.5 gives the modified transformation of the variation as

$$
\begin{aligned}
\widehat{\Delta z}(x,y^{+}) &= x(x,y^{+}) + iy(x,y^{+}) + \big[x(a,b) + iy(a,b) - \big(x(x,y^{+}) + iy(x,y^{+})\big)\big] \\
&= x + i\big(\widehat{\infty} - y^{+}\big) + \big\{a + i\big(\widehat{\infty} - b\big) - \big[x + i\big(\widehat{\infty} - y^{+}\big)\big]\big\} \\
&= x + i\big(\widehat{\infty} - y^{+}\big) + a - x + i\big[\big(\widehat{\infty} - b\big) - \big(\widehat{\infty} - y^{+}\big)\big] \\
&= x + i\widehat{\infty} - iy^{+} + a - x + i\big(y^{+} - b\big) \\
&= \big(x + a - x\big) - i\big(y^{+} + b - y^{+}\big) + i\widehat{\infty} \\
&= \Delta x - i\Delta y^{+} + i\widehat{\infty} \quad .
\end{aligned}
$$

For $\widehat{\Delta z}(x,y^{-})$, we have two conversion functions

$$
x(x,y^{-}) = x \quad , \qquad \text{and} \qquad y(x,y^{-}) = y^{-} - \widehat{\infty} \quad .
$$

Definition 2.4.5 gives the modified transformation of the variation as

$$
\begin{aligned}
\widehat{\Delta z}(x,y^{-}) &= x(x,y^{+}) + iy(x,y^{+}) + \big[x(a,b) + iy(a,b) - \big(x(x,y^{+}) + iy(x,y^{+})\big)\big] \\
&= x + i\big(y^{-} - \widehat{\infty}\big) + \big\{a + i\big(b - \widehat{\infty}\big) - \big[x + i\big(y^{-} - \widehat{\infty}\big)\big]\big\} \\
&= x + i\big(y^{-} - \widehat{\infty}\big) + a - x + i\big[\big(b - \widehat{\infty}\big) - \big(y^{-} - \widehat{\infty}\big)\big] \\
&= x + iy^{-} - i\widehat{\infty} + a - x + i\big(b - y^{-}\big) \\
&= \big(x + a - x\big) + i\big(y^{-} + b - y^{-}\big) - i\widehat{\infty} \\
&= \Delta x + i\Delta y^{-} - i\widehat{\infty} \quad .
\end{aligned}
$$

Remark 2.4.14 Altogether we have

$$\begin{aligned}
\Delta z^{+}(x,y) &= \Delta x - i\Delta y \\
\Delta z^{-}(x,y) &= \Delta x - i\Delta y^{-} \\
\Delta z(x,y^{+}) &= \Delta x - i\Delta y^{+} \\
\Delta z(x,y^{-}) &= \Delta x + i\Delta y^{-} \quad .
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\Delta z}^{+}(x,y) &= \Delta x - i\Delta y + i\widehat{\infty} \\
\widehat{\Delta z}^{-}(x,y) &= \Delta x - i\Delta y - i\widehat{\infty} \\
\widehat{\Delta z}(x,y^{+}) &= \Delta x - i\Delta y^{+} + i\widehat{\infty} \\
\widehat{\Delta z}(x,y^{-}) &= \Delta x + i\Delta y^{-} - i\widehat{\infty} \quad .
\end{aligned}$$

Notice that the exceptional behavior "$+\,i\Delta y^{-}$" observed in the canonical transformation from the Cartesian representation to z^{-} is preserved in the modified transformation.

Remark 2.4.15 Usually there is some freedom to define a variation as either of

$$\Delta x = x + \big(h - x\big) \quad , \qquad \text{or} \qquad \Delta x^{\dagger} = \big(x + h\big) - x \quad .$$

Since Axiom 1.3.19 gives non-associative additive compositions between $\widehat{\mathbb{R}}$-valued quantities, the usual equality between these two expressions is not preserved in the present context . Therefore, we should transform the modified variation with the other bracketing. (The canonical transformation law given by Definition 2.3.3 does not decompose the variation Δx into its parts so the associativity or non-associativity is irrelevant for the canonical transformation.) Before we demonstrate the alternative bracketing, however, it must be noted that it is usually the convention that the variation of x adds a small quantity to $+x$ as in

$$\Delta x = x + \big(h - x\big) \quad ,$$

rather than a large quantity to $-x$ as in

$$\Delta x^{\dagger} = -x + \big(h + x\big) \quad .$$

(The "size" of the quantity $h \pm x$ follows from the specification in Example 2.4.8 that x and h satisfy $0 < x < h$.) In the alternative bracketing denoted by the dagger symbol, we have for ${}'\widehat{\Delta z}^{+}(x,y)$ two conversion functions

$$x(x,y) = x \quad , \qquad \text{and} \qquad y^{+}(x,y) = \widehat{\infty} - y \quad ,$$

such that

$$^{\dagger}\widehat{\Delta z}^{+}(x,y) = \big(z^{+} + h^{+}\big) - z^{+}$$

$$
\begin{aligned}
&= \big(z^+(x,y^+)+h^+\big) - z^+(x,y^+) \\
&= \big(x(x,y)+iy^+(x,y)+a^+ + ib^+\big) - \big(x(x,y)+iy^+(x,y)\big) \\
&= \big[x+i\big(\widehat{\infty}-y\big)+a+i\big(\widehat{\infty}-b\big)\big] - \big[x+i\big(\widehat{\infty}-y\big)\big] \\
&= x+a-x+i\big[\widehat{\infty}-\big(y+b\big)\big] - i\big(\widehat{\infty}-y\big) \\
&= \big(x+a-x\big)+i\big(y-b+y\big) \\
&= \Delta x - i\Delta y \quad .
\end{aligned}
$$

Comparing to Remark 2.4.14, this is exactly the same as the canonically transformed variation $\Delta z^+(x,y)$. For the odd case of ${}^{\dagger}\widehat{\Delta z}(x,y^-)$ we have two conversion functions

$$
x(x,y^-) = x \quad , \qquad \text{and} \qquad y(x,y^-) = y^- - \widehat{\infty} \quad .
$$

so that

$$
\begin{aligned}
{}^{\dagger}\widehat{\Delta z}(x,y^-) &= \big(z+h\big) - z \\
&= \big(z(x,y)+h\big) - z(x,y) \\
&= \big(x(x,y^-)+iy(x,y^-)+a+ib\big) - \big(x(x,y^-)+iy(x,y^-)\big) \\
&= \big[x+i\big(y-\widehat{\infty}\big)+a+i\big(b-\widehat{\infty}\big)\big] - \big[x+i\big(y-\widehat{\infty}\big)\big] \\
&= x+a-x+i\big[\big(y+b\big)-\widehat{\infty}\big] - i\big(y-\widehat{\infty}\big) \\
&= \big(x+a-x\big)+i\big(y+b-y\big) \\
&= \Delta x + i\Delta y \quad .
\end{aligned}
$$

Again, this gives exactly the canonical transformation $\Delta z(x,y^-)$. A similar demonstration shows that all of the modified variations exactly replicate their canonical counterparts in the alternative bracketing.

Theorem 2.4.16 *For functions of the form $f(z)=cz^n$ with $z\in\mathbb{C}$, $n\in\mathbb{R}$, $n\geq 1$, and c being any constant, the modified representational variation always produces the correct derivative $\widehat{d}/dz\, f(z^\pm) = d/dz\, f(z^\pm)$. (Derivative with respect to the Cartesian representation.)*

Proof. For the case of z^+, we have

$$
z^+(x,y^+) = x+iy^+ \quad ,
$$

with two conversion functions

$$
x(x,y) = x \quad , \qquad \text{and} \qquad y^+(x,y) = \widehat{\infty} - y \quad .
$$

As summarized in Remark 2.4.14, the transformation law for the modified variation is

$$
\widehat{\Delta z}^{+}(x,y) = \Delta x - i\Delta y + i\widehat{\infty}
$$

Therefore,

$$\begin{aligned}\frac{\widehat{d}}{dz}\,c\big(z^+\big)^n &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{c\big(z^+ + \widehat{\Delta z}^+(x,y)\big)^n - c\big(z^+\big)^n}{\widehat{\Delta z}^+(x,y)}\\ &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{c\big(x+iy^+ + \Delta x - i\Delta y + i\widehat{\infty}\big)^n - c\big(x+iy^+\big)^n}{\Delta x - i\Delta y + i\widehat{\infty}}\\ &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{c\big[x+i\big(\widehat{\infty}-y\big)+\Delta x - i\Delta y + i\widehat{\infty}\big]^n - c\big[x+i\big(\widehat{\infty}-y\big)\big]^n}{\Delta x - i\Delta y + i\widehat{\infty}}\\ &= \lim_{\Delta x\to 0}\frac{c\big[x+i\big(\widehat{\infty}-y\big)+\Delta x + i\widehat{\infty}\big]^n - c\big[x+i\big(\widehat{\infty}-y\big)\big]^n}{\Delta x + i\widehat{\infty}}\end{aligned}$$

If we take the limit $\Delta x \to 0$, we will divide by $\widehat{\infty}$ and it is obvious that the numerator will also depend on infinity. To motivate L'Hôpital's rule, we will explicitly compute the numerator in the limit $\Delta x \to 0$ with the binomial theorem

$$\big(\xi+\varphi\big)^n = \sum_{k=0}^{n}\binom{n}{k}\xi^{n-k}\varphi^k = \xi^n + \sum_{k=1}^{n}\binom{n}{k}\xi^{n-k}\varphi^k \quad .$$

To condense the relevant expression, let

$$A = x + i\big(\widehat{\infty}-y\big) \quad ,$$

so that

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{c\big(A+\Delta x + i\widehat{\infty}\big)^n - cA^n}{\Delta x + i\widehat{\infty}} &= \frac{c\big(A+i\widehat{\infty}\big)^n - cA^n}{i\widehat{\infty}}\\ &= \frac{c}{i\widehat{\infty}}\left[A^n + \sum_{k=1}^{n}\binom{n}{k}A^{n-k}\big(i\widehat{\infty}\big)^k - A^n\right]\\ &= c\sum_{k=1}^{n}\binom{n}{k}A^{n-k}\frac{\big(i\widehat{\infty}\big)^k}{i\widehat{\infty}} \quad .\end{aligned}$$

Having obtained an expression of the form ∞/∞, we are motivated to use L'Hôpitals' rule to compute the limit in the expression for $\widehat{d}/dz^+\big(cz^n\big)$. It is for this reason that the derivative works for $\gamma=\widehat{\infty}$ even though it fails in the case of finite γ, as in Example 2.4.10. Application of L'Hôpitals' rule yields

$$\frac{\widehat{d}}{dz}\,c\big(z^+\big)^n \overset{*}{=} \lim_{\Delta x\to 0}\frac{\dfrac{d}{d\Delta x}\Big\{c\big[x+i\big(\widehat{\infty}-y\big)+\Delta x + i\widehat{\infty}\big]^n - c\big[x+i\big(\widehat{\infty}-y\big)\big]^n\Big\}}{\dfrac{d}{d\Delta x}\Big(\Delta x + i\widehat{\infty}\Big)} \quad .$$

To avoid any circular reasoning, we use explicitly take the derivatives with respect to Δx using the canonical variation. Taking the derivatives yields

$$\begin{aligned}\frac{\widehat{d}}{dz}c\big(z^+\big)^n &= \lim_{\Delta x\to 0} cn\big[x+i\big(\widehat{\infty}-y\big)+\Delta x+i\widehat{\infty}\big]^{n-1}\\&= cn\big\{x+i\big[\big(\widehat{\infty}-y\big)+\widehat{\infty}\big]\big\}^{n-1}\\&= cn\big[x+i\big[\big(\widehat{\infty}-y\big)\big]^{n-1}\quad .\end{aligned}$$

We have obtained the expression for the derivative of $f(z^+)$ in the Cartesian representation so we should recast it into the z^+ representation with the conversion functions $x=x$ and $y^+=\widehat{\infty}-y$. This gives

$$\begin{aligned}\frac{\widehat{d}}{dz}c\big(z^+\big)^n &= cn\big(x+iy^+\big)^{n-1}\\&= cn\big(z^+\big)^{n-1}\quad .\end{aligned}$$

This is the correct derivative so the theorem is proven for $f(z^+)$. For the case of $f(z^-)$, we have

$$z^-(x,y^-)=x-iy^-\quad ,$$

with two conversion functions

$$x(x,y)=x\quad ,\qquad\text{and}\qquad y^-(x,y)=\widehat{\infty}+y\quad .$$

The transformation law for the modified variation (Remark 2.4.14) is

$$\widehat{\Delta z}^-(x,y)=\Delta x-i\Delta y-i\widehat{\infty}$$

Therefore,

$$\begin{aligned}\frac{\widehat{d}}{dz}c\big(z^-\big)^n &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{c\big(z^-+\widehat{\Delta z}^-(x,y)\big)^n-c\big(z^-\big)^n}{\widehat{\Delta z}^-(x,y)}\\&= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{c\big(x-iy^-+\Delta x-i\Delta y-i\widehat{\infty}\big)^n-c\big(x+iy^-\big)^n}{\Delta x-i\Delta y-i\widehat{\infty}}\\&= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{c\big[x-i\big(\widehat{\infty}+y\big)+\Delta x-i\Delta y-i\widehat{\infty}\big]^n-c\big[x-i\big(\widehat{\infty}+y\big)\big]^n}{\Delta x-i\Delta y-i\widehat{\infty}}\\&= \lim_{\Delta x\to 0}\frac{c\big[x-i\big(\widehat{\infty}+y\big)+\Delta x-i\widehat{\infty}\big]^n-c\big[x-i\big(\widehat{\infty}+y\big)\big]^n}{\Delta x-i\widehat{\infty}}\end{aligned}$$

Following the case of z^+, it is obvious that L'Hôpital's rule is called for. It follows that

$$\frac{\widehat{d}}{dz}c\big(z^-\big)^n\overset{*}{=}\lim_{\Delta x\to 0}\frac{\dfrac{d}{d\Delta x}\Big\{c\big[x-i\big(\widehat{\infty}+y\big)+\Delta x-i\widehat{\infty}\big]^n-c\big[x-i\big(\widehat{\infty}+y\big)\big]^n\Big\}}{\dfrac{d}{d\Delta x}\Big(\Delta x-i\widehat{\infty}\Big)}$$

$$
\begin{aligned}
&= \lim_{\Delta x\to 0} cn\big[x - i\big(\widehat{\infty} + y\big) + \Delta x - i\widehat{\infty}\big]^{n-1} \\
&= cn\big\{x - i\big[\big(\widehat{\infty} + y\big) + \widehat{\infty}\big]\big\}^{n-1} \\
&= cn\big[x - i\big[\big(\widehat{\infty} + y\big)\big]^{n-1} \\
&= cn\big(x - iy^-\big)^{n-1} \\
&= cn\big(z^-\big)^{n-1} \quad .
\end{aligned}
$$

This is the correct derivative so we have proven the theorem in each case of $z^\pm$.

Main Theorem 2.4.17 *If $z \in \mathbb{C}$ and $f(z) = e^z$, then the modified representational variation always produces the correct derivative $\widehat{d}/dz\, f(z^\pm)$.*

Proof. The exponential function is defined as

$$
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad , \qquad \text{so} \qquad \frac{\widehat{d}}{dz} e^{z^\pm} = \sum_{n=0}^{\infty} \frac{\widehat{d}}{dz} \frac{\big(z^\pm\big)^n}{n!} \quad .
$$

For the terms with $n \geq 1$, Theorem 2.4.16 proves that the modified variation will generate the correct derivatives. For the $n = 0$ term which is identically 1, the derivative of a constant vanishes trivially. Since the modified variation only enters the definition of the derivative through $f(z) \to f(z + \widehat{\Delta z})$, the modified variation can never enter into an expression which did not depend on z to begin with. Therefore, we have proven that when $e^{z^\pm}$ is written as an infinite series, the $\widehat{d}/dz$ operator relying on the modified variation will produce the correct derivative for each term. Therefore, $\widehat{d}/dz$ will produce the correct derivative when it acts on $e^{z^\pm}$.

§3 Proof of limits of sine and cosine at infinity

§3.1 Refutation of proof of nonexistence of limits at infinity

Definition 3.1.1 We say that the limit of a sequence exists if and only if all of its subsequences converge to the same value.

Proposition 3.1.2 It is impossible to compute the limits

$$
\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sin(x) \quad , \qquad \text{and} \qquad \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \cos(x) \quad ,
$$

because all of their subsequences do not converge to the same values.

Refutation. Definition 3.1.1 requires that for a limit

$$\lim_{x\to\infty} F(x) = l \quad ,$$

to exist, the function F must converge to l in all of its subsequences. Usually one attempts to prove the present proposition by showing a contradiction: all subsequences do not converge to the same value. To that end, consider two subsequences of x

$$x_n = 2n\pi + \frac{\pi}{2} \quad , \qquad \text{and} \qquad y_n = 2n\pi .$$

For any $n \in \mathbb{N}$, we have

$$\sin(x_n) = 1 \quad , \qquad \text{and} \qquad \sin(y_n) = 0 \quad .$$

It is said, therefore, that it is impossible for all subsequences of $\sin(x)$ to converge to some constant l. However, to refute this proposition, one must note that the convergence of the sequences are determined by the final n points, not the first n points. By Main Theorem 1.3.31, the final n points of an increasing sequence of $\mathbb{R}$ approaching infinity (increasing without bound) will have the form

$$x = \widehat{\infty} - b \quad .$$

Both x_n and y_n have the property

$$\lim_{n\to\infty} x_n = \infty \quad , \qquad \text{and} \qquad \lim_{n\to\infty} y_n = \infty \quad ,$$

so, by Axiom 1.3.2 giving $\|\widehat{\infty}\| = \|\infty\|$, we may write identically

$$\lim_{n\to\widehat{\infty}} x_n = \widehat{\infty} \quad , \qquad \text{and} \qquad \lim_{n\to\widehat{\infty}} y_n = \widehat{\infty} \quad .$$

(Although the x_n and y_n of the present context are exactly in the form of the x_n and y_n used to demonstrate a contradiction in Example 1.3.13, we have not presently replicated the procedure generating the contradiction and the non-contradiction property does not apply.) Since we know each sequence approaches $\widehat{\infty}$, and we know that the points of each sequence are spaced by 2π, we may introduce numbering by n' and m' to write the final n points of x_n and x_m as

$$x_{\widehat{\infty}-n'} = \widehat{\infty} - 2n'\pi \quad , \qquad \text{and} \qquad x_{\widehat{\infty}-m'} = \widehat{\infty} - 2m'\pi \quad ,$$

where

$$n', m' \to 0 \quad , \qquad \text{as} \qquad n, m \to \infty \quad .$$

Since $\widehat{\infty} - n \neq \widehat{\infty}$, all of these points are distinct. It is obvious that both sequences converge to the same value. We have refuted the proposition by showing that the final n terms of the series have a different form than the first n terms. The argument in favor of the proposition does not consider the final n terms at all.

Remark 3.1.3 By considering n near infinity in the refutation of Proposition 3.1.2, that is $n = \widehat{\infty} - n'$, we have not restricted n to $\mathbb{N}$. As mentioned in Remark 1.3.5, the cut definition of $\mathbb{R}$ (Definition 1.1.5) does not preserve Dedekind's least upper bound property because Axiom 1.3.4 states that all $x \in \widehat{\mathbb{R}}$ are greater than every $n \in \mathbb{N}$. If we let the n in the refutation of Proposition 3.1.2 include elements of $\widehat{\mathbb{R}}$ on the final approach to infinity, then for consistency we must also allow such n in the definition of the exponential function used in Main Theorem 2.4.17. When n can exceed $\mathbb{N}$ on approach to infinity, the definition of the exponential becomes

$$e^x = 1 + \sum_{n\in\mathbb{N}} \frac{x^n}{n!} + \{\text{intermediate values}\} + \sum_{n\in\mathbb{N}} \frac{x^{(\widehat{\infty}-n)}}{\big(\widehat{\infty}-n\big)!} \quad .$$

In Theorem 3.1.4 we will prove that the final sum is equal to zero. In the present context, we have not defined numbers which are larger than every element of $\mathbb{N}$ yet less than every element of $\widehat{\mathbb{R}}$ but the existence of such numbers is implied by the connected property of $\mathbb{R}$. (This property is implied directly from Definition 1.1.5 giving all $x \in \mathbb{R}$ as elements of continuum.) The case of the intermediate values is specified in Reference [4] and it follows that all such terms are identically zero. Allowing n to exceed $\mathbb{N}$ does not change the exponential function at all. Theorem 3.1.4 together with an extension to the intermediate values as in Reference [4] prove that

$$e^x = 1 + \sum_{n\in\mathbb{N}} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad .$$

Theorem 3.1.4 *Quotients of the form* $x^{\widehat{\infty}-n}/\big(\widehat{\infty}-n\big)!$ *are identically zero.*

Proof. We have

$$\frac{x^{(\widehat{\infty}-n)}}{\big(\widehat{\infty}-n\big)!} = \frac{x^{\widehat{\infty}}x^{-n}}{\big(\widehat{\infty}-n\big)!} = \frac{\widehat{\infty}}{\big(\widehat{\infty}-n\big)!} = \frac{\widehat{\infty}}{\big(\widehat{\infty}-n\big)} \frac{1}{\big(\widehat{\infty}-n-1\big)!} \quad .$$

Although $\widehat{\infty}/\widehat{\infty}$ is undefined, we can compute $\widehat{\infty}/\big(\widehat{\infty}-b\big)$ as a limit. Using the property of limits that the limit of a product is a product of limits, we find

$$\begin{aligned}\frac{x^{(\widehat{\infty}-n)}}{\big(\widehat{\infty}-n\big)!} &= \lim_{x\to 0} \frac{\frac{1}{x}}{\big(\frac{1}{x}-n\big)} \lim_{x\to 0} \frac{1}{\big(\frac{1}{x}-n-1\big)} \lim_{x\to 0} \frac{1}{\big(\frac{1}{x}-n-2\big)} \cdots \\ &= \lim_{x\to 0} \frac{1}{\big(1-xn\big)} \lim_{x\to 0} \frac{x}{\big(1-xn-x\big)} \lim_{x\to 0} \frac{x}{\big(1-xn-x\big)} \cdots \\ &= 1 \times 0 \times 0 \cdots = 0\end{aligned}$$

All such quotients are identically zero.

§3.2 Proof of limits of sine and cosine at infinity

Theorem 3.2.1 *The values of sine and cosine at infinity are*

$$\sin(\infty) = 0 \quad , \qquad \textit{and} \qquad \cos(\infty) = 1 \quad .$$

Proof. We have proven in Theorem 2.3.21 that

$$\frac{d}{dz_1} e^{z_2} = e^{z_2} \quad .$$

This is the "correct derivative." For $f(z) = e^z$ in the case of

$$\frac{\widehat{d}}{dz} f(z^+) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \widehat{\Delta z}^+) - f(z^+)}{\widehat{\Delta z}^+} \quad ,$$

we have proven in Main Theorem 2.4.17 that the definition of the derivative relying on the modified variation $\widehat{\Delta z}^+$ will produce the correct derivative. We have

$$z^+(x, y^+) = x + iy^+ \quad ,$$

with two conversion functions

$$x(x, y) = x \quad , \qquad \text{and} \qquad y^+(x, y) = \widehat{\infty} - y \quad .$$

The transformation for the modified variation is listed in Remark 2.4.14 as

$$\widehat{\Delta z}^+(x, y) = \Delta x - i\Delta y + i\widehat{\infty} \quad .$$

Evaluation yields

$$\begin{aligned}
\frac{\widehat{d}}{dz} e^{z^+} &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(z^+ + \widehat{\Delta z}^+) - f(z^+)}{\widehat{\Delta z}^+} \\
&= e^{x+iy^+} \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{e^{\Delta x - i\Delta y + i\widehat{\infty}} - 1}{\Delta x - i\Delta y + i\widehat{\infty}} \\
&= e^{x+iy^+} \lim_{\Delta x \to 0} \frac{e^{\Delta x + i\widehat{\infty}} - 1}{\Delta x + i\widehat{\infty}} \\
&\overset{*}{=} e^{x+iy^+} \lim_{\Delta x \to 0} e^{\Delta x + i\widehat{\infty}} \\
&= e^{x+iy^+} e^{i\widehat{\infty}} \\
&= e^{z^+} e^{i\widehat{\infty}} \quad .
\end{aligned}$$

The exponential function is an eigenfunction of the derivative with unit eigenvalue so

$$1 = e^{i\widehat{\infty}} = \cos(\widehat{\infty}) + i\sin(\widehat{\infty}) \quad .$$

Equating the real and imaginary parts gives

$$\sin(\widehat{\infty}) = 0 \quad , \qquad \text{and} \qquad \cos(\widehat{\infty}) = 1 \quad .$$

Theorem is proven with Axiom 1.3.2:

$$\sin(\widehat{\infty}) = \sin(\infty) \quad , \qquad \text{and} \qquad \cos(\widehat{\infty}) = \cos(\infty) \quad .$$

Main Theorem 3.2.2 *The limits of sine and cosine at infinity are*

$$\lim_{x\to\infty} \sin(x) = 0 \quad , \qquad \textit{and} \qquad \lim_{x\to\infty} \cos(x) = 1 \quad .$$

Proof. A function has a limit l if and only if the function converges to l in any subsequence. Since the set of $x \in \mathbb{R}$ converges to ∞, all of its subsequences also converge to ∞. Therefore, for any subsequence x_n of x, we have

$$\lim_{n\to\infty} \sin(x_n) = \sin(\infty) \quad , \qquad \text{and} \qquad \lim_{n\to\infty} \cos(x_n) = \cos(\infty) \quad .$$

Proof follows from Theorem 3.2.1: $\sin(\infty) = 0$ and $\cos(\infty) = 1$.

Theorem 3.2.3 *Sine and cosine are continuous at infinity.*

Proof. We say that a function is continuous at a point if

$$\lim_{x\to x_0} f(x) = f(x_0) \quad .$$

Sine and cosine are such that

$$\lim_{x\to\infty} \sin(x) = \sin(\infty) \quad , \qquad \text{and} \qquad \lim_{x\to\infty} \cos(x) = \cos(\infty) \quad .$$

Both functions are continuous at infinity.

Theorem 3.2.4 *The values of sine and cosine at ∞ preserve the odd- and evenness of sine and cosine respectively.*

Proof. We have

$$z^-(x, y^-) = x - iy^- \quad ,$$

with two conversion functions

$$x(x, y) = x \quad , \qquad \text{and} \qquad y^-(x, y) = \widehat{\infty} + y \quad .$$

Remark 2.4.14 gives the relevant modified variation as

$$\widehat{\Delta z}^-(x, y) = \Delta x - i\Delta y - i\widehat{\infty} \quad ,$$

so we have

$$\begin{aligned}
\frac{\widehat{d}}{dz}\,e^{z^-} &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(z^- + \widehat{\Delta z}^{\,-}) - f(z^-)}{\widehat{\Delta z}^{\,-}} \\
&= e^{x-iy^-} \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{\Delta x - i\Delta y - i\widehat{\infty}} - 1}{\Delta x - i\Delta y - i\widehat{\infty}} \\
&= e^{x-iy^-} \lim_{\Delta x\to 0} \frac{e^{\Delta x - i\widehat{\infty}} - 1}{\Delta x + i\widehat{\infty}} \\
&\overset{*}{=} e^{x-iy^-} \lim_{\Delta x\to 0} e^{\Delta x - i\widehat{\infty}} \\
&= e^{x-iy^-} e^{-i\widehat{\infty}} \\
&= e^{z^-} e^{-i\widehat{\infty}} \quad .
\end{aligned}$$

It follows that

$$1 = e^{-i\widehat{\infty}} = \cos(-\widehat{\infty}) + i\sin(-\widehat{\infty}) \quad .$$

Equating the real and imaginary parts gives

$$\cos(-\widehat{\infty}) = 1 \quad , \qquad \text{and} \qquad \sin(-\widehat{\infty}) = 0 \quad .$$

Therefore,

$$\cos(-\widehat{\infty}) = \cos(\widehat{\infty}) \quad , \qquad \text{and} \qquad \sin(-\widehat{\infty}) = -\sin(\widehat{\infty}) \quad .$$

Sine is an odd function and cosine is an even function.

Theorem 3.2.5 *Sine and cosine satisfy the double angle identities at infinity.*

Proof. The relevant identities are

$$\sin(2x) = 2\sin(x)\cos(x) \quad , \qquad \text{and} \qquad \cos(2x) = 1 - \sin^2(x) \quad .$$

These identities are satisfied trivially for $x = \widehat{\infty}$.

References

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[4] Jonathan W. Tooker. Fractional Distance: The Topology of the Real Number Line with Applications to the Riemann Hypothesis. *viXra:1906.0237*, (2019).