On the Infinite Product for the Ratio of k-th Power and Factorial

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And Jesus said unto them, I am the bread of life: he that cometh to me shall never hunger; and he that believe on me shall never thirst. John 6:35.

ABSTRACT. I derive an infinite product for the ratio of k-th power and factorial.

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1. INTRODUCTION

In present paper, I derive the following infinite products:

$$\frac{z^k}{k!} = \prod_{j=1}^{\infty} \left(1+\frac{k}{j}\right)\left(1-\frac{1}{j+z}\right)^k,$$

and, obviously,

$$z = \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)\left(1-\frac{1}{j+z}\right).$$

2. THE MAIN THEOREM

2.1. The Infinite Product for the Ratio of K-th Power and Factorial.

Theorem 2.1. *If $z \in \mathbb{C}$ and $k \in \mathbb{Z}^+$, then*

$$\frac{z^k}{k!} = \prod_{j=1}^{\infty} \left(1+\frac{k}{j}\right)\left(1-\frac{1}{j+z}\right)^k, \tag{2.1}$$

where $k!$ denotes the factorial.

Proof. I well know the finite product identity

$$\frac{z^k}{k!} = \prod_{r=1}^{k} \left(\frac{z}{r}\right). \tag{2.2}$$

On the other hand, I have the infinite product representation [1, Lemma 1, p. 2]

$$\frac{a}{b}=\prod_{j=1}^{\infty}\frac{(a+j-1)(b+j)}{(a+j)(b+j-1)}. \tag{2.3}$$

Replace a by z and b by r in (2.3) and encounter

$$\frac{z}{r}=\prod_{j=1}^{\infty}\frac{(z+j-1)(r+j)}{(z+j)(r+j-1)}. \tag{2.4}$$

From (2.2) and (2.4), it follows that

$$\begin{aligned}\frac{z^k}{k!}&=\prod_{r=1}^{k}\prod_{j=1}^{\infty}\frac{(z+j-1)(r+j)}{(z+j)(r+j-1)}\\&=\prod_{j=1}^{\infty}\prod_{r=1}^{k}\frac{(z+j-1)(r+j)}{(z+j)(r+j-1)}\\&=\prod_{j=1}^{\infty}\left(1+\frac{k}{j}\right)\left(1-\frac{1}{j+z}\right)^k,\end{aligned}$$

which is the desired result. □

2.2. The Infinite Products for the K-th Power and the z.

Theorem 2.2. *If $z\in\mathbb{C}$ and $k\in\mathbb{Z}^+$, then*

$$z^k=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^k\left(1-\frac{1}{j+z}\right)^k, \tag{2.5}$$

where z^k denotes the k-th power of z.

Proof. I well know the finite product identity

$$\frac{z^k}{k}=\frac{z^k(k-1)!}{k!}=\frac{z^k}{k!}\cdot\Gamma(k). \tag{2.6}$$

On the other hand, I know the Euler's infinite product representation for gamma function [1, (1), p. 1]

$$\Gamma(k)=\frac{1}{k}\prod_{j=1}^{\infty}\frac{\left(1+\frac{1}{j}\right)^k}{\left(1+\frac{k}{j}\right)}. \tag{2.7}$$

From Theorem 2.1, (2.6) and (2.7), I conclude that

$$\begin{aligned}\frac{z^k}{k}&=\frac{1}{k}\prod_{j=1}^{\infty}\left(1+\frac{k}{j}\right)\left(1-\frac{1}{j+z}\right)^k\cdot\frac{\left(1+\frac{1}{j}\right)^k}{\left(1+\frac{k}{j}\right)}\\ \Rightarrow z^k&=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^k\left(1-\frac{1}{j+z}\right)^k,\end{aligned} \tag{2.8}$$

which is the desired result. □

Corollary 2.3. *If $z \in \mathbb{C}$, then*

$$z = \prod_{j=1}^{\infty} \left(1 + \frac{1}{j}\right)\left(1 - \frac{1}{j+z}\right). \tag{2.9}$$

Proof. Set $k = 1$ in the Theorem 2.2. This gives the desired result. □

3. Exercises

Exercise 3.1. Prove that

$$c\,{}_2F_1(a,b;c;z) + {}_2F_1(a,b;c+1;z) = {}_3F_2(2,a,b;1,c+1;z) + c\,{}_2F_1(a,b;c+1;z).$$

Exercise 3.2. Prove that

$$\frac{1}{(1+z)^k \Gamma(1-k)} = \prod_{j=1}^{\infty} \left(1 - \frac{k}{j}\right)\left(1 + \frac{1}{j+z}\right)^k.$$

Reference

[1] Guedes, Ediglles, *Infinite Product Representations for Binomial Coefficient, Pochhammer's Symbol, Newton's Binomial and Exponential Function*, June 27, 2016, **viXra:1611.0049**.