Physically Consistent Probability Density in Noncommutative Quantum Mechanics

Andrey Gomes Martins [a], Ronaldo Vieira Lobato [b†], Geanderson Araújo Carvalho [b] and Antônio de Passos Neto Cronemberger Galvão[a].
[a] *Universidade do Estado do Pará,*
Av: Djalma Dutra s/n, 66130-200, Telégrafo, Belém, PA, Brazil.
[b] *Instituto Tecnológico de Aeronáutica,*
Praça Marechal Eduardo Gomes 50, 12228-900 Vila das Acácias, São José dos Campos, SP, Brazil.
[†] Email: rvlobato@ita.br

1 Introduction

Some time ago a result obtained in String Theory [1] has lead some field theorists to study a deformed version of Quantum Electrodynamics [2], as well as other field theories [3]. Since then it has been generally assumed that such theories could be relevant to the understanding of physics at the Planck scale [4].

In the context of Noncommutative Quantum Theories the ordinary spatial coordinates x_i are replaced by a set of operators, the noncommutative coordinates $\hat{x}_i$, satisfying a commutation relation of the form:

$$[\hat{x}_i, \hat{x}_j] = i\Theta_{ij} \,, \tag{1}$$

where Θ_{ij} is a constant, real-valued, antisymmetric matrix.

2 Quantum Mechanics on a Noncommutative Plane

Geometry: The noncommutativity of the Moyal plane is described by the commutation relation [5]

$$\hat{x}\hat{y} - \hat{y}\hat{x} = i\theta \,, \tag{2}$$

where θ is a real parameter.

With the aid of Fourier analysis the elements of the noncommutative plane can be written in terms of the noncommutative coordinates ($\hat{x}_i$) as:

$$\hat{\psi} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}(\vec{k})\, e^{i(k_1\hat{x}+k_2\hat{y})} \,, \tag{3}$$

where $\tilde{\psi}(\vec{k})$ is the Fourier transform of the complex valued function

$$\psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\psi}(\vec{k})\, e^{i(k_1 x + k_2 y)}\,, \tag{4}$$

Kinematics:

The action of the noncommutative momenta $\hat{p}_i$ on wave functions is defined by

$$\hat{p}_1(\hat{\psi}) = \frac{\hbar}{\theta}[\hat{y}\,,\,\hat{\psi}] \quad , \quad \hat{p}_2(\hat{\psi}) = -\frac{\hbar}{\theta}[\hat{x}\,,\,\hat{\psi}]\,. \tag{5}$$

Dynamics: Time evolution is governed by the Algebraic Schrödinger Equation [6],

$$\left[\frac{(\hat{p}_1^2 + \hat{p}_2^2)}{2m} + V(\hat{x},\hat{y})\right]\hat{\psi} = i\hbar\frac{\partial\hat{\psi}}{\partial t}\,. \tag{6}$$

2.1 Weyl Correspondence

The Weyl correspondence ($\mathcal{W}$) maps the noncommutative element $\hat{\psi}$ of Eq. (3) into its ordinary counterpart $\psi(x,y)$, given by Eq. (4):

$$\hat{\psi} \overset{\mathcal{W}}{\longmapsto} \psi\,. \tag{7}$$

The product $\hat{\psi}\hat{\phi}$ gets mapped into the Moyal product [7], that is:

$$\hat{\psi}\hat{\phi} \overset{\mathcal{W}}{\longmapsto} (\psi \star \phi)\,, \tag{8}$$

where

$$(\psi\star\phi)(x,y) \equiv \psi(x,y)\, e^{\frac{i\theta}{2}(\overleftarrow{\partial_x}\,\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\,\overrightarrow{\partial_x})}\,\phi(x,y)\,, \tag{9}$$

or equivalently, by expanding the operatorial exponential in Taylor series,

$$(\psi\star\phi)(x,y) = \sum_{n=0}^{\infty}\left(\frac{i\theta}{2}\right)^n \sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\frac{\partial^{n-k}}{\partial x^{n-k}}\left(\frac{\partial^k\psi}{\partial y^k}\right)\frac{\partial^k}{\partial x^k}\left(\frac{\partial^{n-k}\phi}{\partial y^{n-k}}\right). \tag{10}$$

2.2 The non-commutative Schrödinger equation

Applying the Weyl Correspondence in both sides of Eq. (6) we find the basic differential equation of NCQM [7], that is,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(x,y)\star\psi = i\hbar\frac{\partial\psi}{\partial t}\,. \tag{11}$$

3 The probability density problem

3.1 Probability density in Quantum Mechanics

Let $\psi(x, y, t)$ be the solution of the Schrödinger equation, with a potential energy $V(x, y)$. The ordinary probability density reads:

$$\rho_0(x, y, t) = \psi(x, y, t)\,\overline{\psi(x, y, t)}\,. \tag{12}$$

The probability conservation is guaranteed by the continuity equation, that is,

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot \vec{J}_0 = 0\,, \tag{13}$$

where the probability current density $\vec{J}_0$ reads

$$\vec{J}_0 = \frac{\hbar}{2mi}\left(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}\right). \tag{14}$$

3.2 Probability density in the noncommutative plane

Difficulty: Eq. (12) cannot be interpreted as the probability density of NCQM, since it does not satisfy the continuity equation [8].

In order to try to solve the above problem it seems to be natural to replace the punctual product in Eq. (12) by the Moyal product. There are two nonequivalent possibilities:

$$\rho'_\theta(x, y, t) = \psi(x, y, t) \star \overline{\psi(x, y, t)} \tag{15}$$

and

$$\rho_\theta(x, y, t) = \overline{\psi(x, y, t)} \star \psi(x, y, t)\,. \tag{16}$$

Difficulties: Definitions (15) and (16) are not positive-definite. Besides, the formula (15) does not satisfy the continuity equation.

In spite of the above difficulty, it is crucial to note that Eq. (16) fulfills the continuity equation, that is,

$$\frac{\partial \rho_\theta}{\partial t} + \nabla \cdot \vec{J}_\theta = 0\,, \tag{17}$$

where

$$\vec{J}_\theta = \frac{\hbar}{2mi}\left(\overline{\psi} \star \nabla\psi - \nabla\overline{\psi} \star \psi\right)\,. \tag{18}$$

3.3 Effective probability densities

The commutation relation (2) leads to a Heisenberg-like uncertainty relation, that is,

$$\Delta x \Delta y \geq \frac{|\theta|}{2}\,, \tag{19}$$

so that the exact knowledge of x prohibits the knowledge of y, and *vice versa*. In view of the above problem we propose to look for a probability density which is a function of just one of the spatial coordinates. So we introduce

$$\rho_1(x) = \int_{-\infty}^{\infty} \overline{\psi} \star \psi \, dy \quad ; \quad \rho_2(y) = \int_{-\infty}^{\infty} \overline{\psi} \star \psi \, dx\,. \tag{20}$$

By integrating Eq. (17) we can show that Eq. (20) satisfy the one-dimensional continuity equation, that is:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial j_1}{\partial x} = 0 \quad ; \quad \frac{\partial \rho_2}{\partial t} + \frac{\partial j_2}{\partial x} = 0\,, \tag{21}$$

where

$$j_1(x,t) = \int_{-\infty}^{\infty} J_x(x,y,t)\, dy, \qquad \text{and} \qquad j_2(y,t) = \int_{-\infty}^{\infty} J_y(x,y,t)\, dx\,, \tag{22}$$

where J_x and J_y are the Cartesian components of $\vec{J}$.
Besides, one can also show that Eq. (20) are both positive definite, so that ρ_i can be interpreted as the physical probability densities of NCQM. It is also important to notice that all the results of this work can be generalized to higher dimensions.

4 Results and conclusions

The definitions proposed in this work, Eq. (20) and Eq. (22), can be respectively interpreted as the probability densities and probability currents associated to the measures of the non-commutative spatial coordinates, since they are compatible with the conservation of probability and positive definite. The formalism developed can be useful to the formulation of the boundary value problem in NCQM.

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