An attempt to find the logical place for a non-Goldbach even number

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With the use of the even/odd number analysis it was found that there is no non-Goldbach even integer > 4 number possible.

INTRODUCTION

In this paper we look at Goldbach's binary conjecture,*every even number* > 4 *is the sum of two* > 1 *odd prime numbers* [1]. This is one of the eldest and best known unsolved problems in mathematics. This conjecture is also called the strong conjecture. We will only be concerned in this paper with the strong conjecture. The approach to the conjecture such as described below also touches upon aspects of philosophy of mathematics.

In the paper the postulate is the existence of, at least one, even integer number that is not the sum of two prime numbers. This is called the non-Goldbach even integer number. The point raised in this paper is that if one postulates the possibility that there exists a non Goldbach even number, then *where* can we find it.

Many have worked along different lines of thought on Goldbach's strong, or binary, conjecture. The paper refers to additive prime number theory [1], the work of Vinogradov [2] and van der Corput [3]. Van der Corput's approach seems to partly build on Landau's work [4]. The latter work from 1909 holds a wealth of history of prime number theory. Furthermore, in 1938 Pipping [5] verified the conjecture for even numbers $n \leq 10^5$. This is long before the existence of a desk computer. The even numbers > 4 that obey Goldbach's definition of being the sum of two odd, > 1, prime numbers, are called Goldbach numbers. If Goldbach's conjecture is not true, there will be at least one non-Goldbach number.

The present paper is in a certain sense the extension of the work of S. Marhall [6]. It is demonstrated here that a non-Goldbach number cannot be construed without running into a contradiction.

THE SEARCH: WHERE IS THE NON-GOLDBACH NUMBER

In our paper we will look at the problem from the viewpoint of even - odd partitioning of integer numbers. Let us, therefore at the start of the paper define two subsets, $\mathbb{E}$ and $\mathbb{O}$, of the positive integer numbers, $\mathbb{N} \equiv \{0, 1, 2, 3, \dots\}$. Therefore,

$$\begin{aligned} \mathbb{E} &= \{\xi \in \mathbb{N} \,|\, \xi = 2n, \,\&\, n \in \mathbb{N}\} = \{0, 2, 4, 6, \dots\} = 2\mathbb{N} \\ \mathbb{O} &= \{\xi \in \mathbb{N} \,|\, \xi = 2n + 1, \,\&\, n \in \mathbb{N}\} = \{1, 3, 5, \dots\} = \mathbb{E} + 1 \end{aligned} \tag{1}$$

The $\mathcal{N}_x = \mathbb{N} \backslash \{0, 1, , \dots [x]\}$ containss the positive integer numbers greater than x. Here, $[x]$ is the first integer number $\leq$ x. E.g. $[3.54] = 3 \in \mathbb{N}$ and $[3] = 3$. Notation convention of [9] is employed.

Furthermore, $\mathbb{P}$ is the set of prime numbers in $\mathbb{O}$, excluding $p = 1$. Our set $\mathbb{P}$ contains prime numbers that are by definition a natural number greater than 2 and can not be divided, leaving rest zero, by all integer numbers that are > 1 and $< p$, [4]. In [4] we also find a simple proof that there are (countably) infinite many prime numbers, $\mathbb{P} \subset \mathbb{O}$.

Preliminary definition of crucial sets.

Subsequently, let us define the following set, for arbitrary, $n \in \mathcal{N}_3 = \{4, 5, 6, \dots\}$

$$\mathcal{A}_{2n} = \{\alpha \in \mathbb{O} \,|\, p \in \mathbb{P}, 2 < p \leq 2n, n \in \mathcal{N}_3, \, \alpha = 2n - p\} \tag{2}$$

Note that $\mathcal{A}_{2n} \subset \mathbb{O}$. This is true because an $\mathbb{E}$ number minus an $\mathbb{O}$ number gives an $\mathbb{O}$ number. The next set to be defined is one in which a second prime number $p^* \in \mathbb{P}$ is used in the definition. The * indicator is to make the difference between the (generating) prime numbers in the set and an arbitrary prime outside the set.

So we can derive from (2) a set for arbitrary $p^* \in \mathbb{P}$

$$\mathcal{A}_{2n-p^*} = \{\alpha \in \mathbb{E} \,|\, p \in \mathbb{P}, p^* \in \mathbb{P}, n \in \mathcal{N}_3, (\alpha = 2n - p^* - p) \,\&\, (2 < p \leq 2n - p^*)\} \tag{3}$$

This set $\mathcal{A}_{2n-p^*}$ is a subset of $\mathbb{E}$. Hence $k \in \mathbb{N}$ such that $2k \in \mathcal{A}_{2n-p^*}$ is possible. Note also that for a given set $\mathcal{A}_{2n-p^*}$, n and p^* are fixed and $p \leq 2n - p^*$ varies. The right hand side of the requirement in (3), i.e. $2n - p^*$ is an $\mathbb{O}$ number. We then have, $2k = 2n - p^* - p$. The (n, p^*) are in a sense the coordinates of the set. The (n, p^*) coordinates are not unique. We have, e.g. $\mathcal{A}_{60-59} = \mathcal{A}_{6-5}$. So, the coordinates $(60, 59)$ and $(6, 5)$ refer to the same set. We note in advance that when $2n - p^* < 0$ then $\mathcal{A}_{2n-p^*} = \emptyset$.

As an example let us look at $2n = 46$ and $p^* = 23$. We have $(p \leq 2n-p^* = 46-23) \Rightarrow p \in \{23, 19, 17, 13, 11, 7, 5, 3\}$, hence, $\mathcal{A}_{46-23} = \{0, 4, 6, 10, 12, 16, 18, 20\}$. The role of p^* and p can be interchanged. Let us briefly look at $\mathcal{A}_{46-19}$. Here $p^* = 19$ and we can have $p = 23$ for $2 < 23 \leq 46 - 19 = 27$, leading to $2k = 46 - 19 - 23 = 4$.

Two other examples are $\mathcal{A}_{60-23} = \{0, 6, 8, 14, 18, 20, 24, 26, 30, 32, 34\}$ and $\mathcal{A}_{38-31} = \{0, 2, 4\}$. If we join $\mathcal{A}_{46-23}$ with $\mathcal{A}_{60-23}$ and e.g. $\mathcal{A}_{38-31}$, we obtain the first, 11, $\mathbb{E}$ numbers, or, $\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\} \cup \{...\}$.

Note that the lowest value of n is 4 in $\mathcal{N}_3$. We are in that case looking at $2 < p \leq 8 - p^*$. Hence, because $8 > p^*$, we have $p^* \in \{7, 5, 3\}$. Hence, when $p^* = 7$ the $\mathcal{A}$ is empty. For $p^* = 3$, we find, $2k = 8-3-5 = 0$, or, $2k = 8-3-3 = 2$, i.e. $\mathcal{A}_{8-3} = \{0, 2\}$. For $p^* = 5$ we find, $p = 3$ as the only possibility. Hence, $\mathcal{A}_{8-5} = \{0\}$.

Let us subsequently define an integer number $N(n, k \,|\, p^*, p) > 2$. Then we can derive from $2k \in \mathcal{A}_{2n-p^*}$ that $2N(n, k \,|\, p^*, p) = 2(n - k) = p^* + p$. The idea is to have arbitrary $n \in \mathcal{N}_3$ and to match a $k \in \mathbb{N}$ such that $2N(n, k \,|\, p^*, p)$ is a Goldbach even number. Such a number can be written as the sum of two primes. Let us call the $k \in \mathbb{N}$, i.e. $0 \leq k \in \mathbb{N}$ and $k < \infty$, the (finite) Goldbach number generator. In the subsequent sections we will concentrate on k to investigate if there is a k as part of a non-Goldbach number, i.e. $2N(n, k)$ not the sum of any two prime numbers from $\mathbb{P}$.

The Goldbach & non-Goldbach numbers

Let us for clarity repeat the definitions. A Goldbach number is a positive even integer that *can* be expressed as the sum of two odd primes. The set of Goldbachs is denoted with $\mathcal{G}$. A non-Goldbach number is a positive even integer that *cannot* be expressed as the sum of two odd primes. The demonstration of the existence of non-Goldbach numbers would invalidate the Goldbach conjecture. The set of non-Goldbachs is denoted with $\mathcal{G}_\neg$

If we want to prove that all even numbers are Goldbach numbers, we must demonstrate that the generators k exhaust the join of all $\mathcal{A}_{...}$ sets into $\mathbb{E}$. I.e.

$$\mathcal{E} \equiv \cup_{n \in \mathcal{N}_3} \cup_{p^* \in \mathbb{P}} \mathcal{A}_{2n-p^*} \tag{4}$$

and $\mathcal{E} = \mathbb{E}$. Below we will prove that $\mathcal{E} = \mathbb{E}$. This entails that $\mathcal{G}_\neg = \emptyset$.

There are infinite prime numbers in $\mathbb{P}$. There are unknown primes. When $\mathcal{E} = \mathbb{E}$, the expression in (4) embraces all the Goldbach numbers. Any pair of $(n, k)|_{n>k}$, with $2k \in \cup_{n \in \mathcal{N}_3} \cup_{p^* \in \mathbb{P}} \mathcal{A}_{2n-p^*} = \mathcal{E}$, gives $k \in \{0, 1, 2, 3, \ldots\} = \mathbb{N}$. Moreover, $(n, k)|_{n>k}$ then has associated to it at least one proper pair of $(p^*, p) \in \mathbb{P}^2$. In turn, $(n, k)|_{n>k}$ can be associated to $N(n, k|p^*, p) = n - k \in \mathbb{N}$. This number can then be employed for constructing $2N(n, k \,|\, p^*, p) = 2(n - k) = p^* + p > 0$. E.g. $7 + 3 = 2(5 - 0) = 2(6 - 1) = 2(7 - 2) = \ldots$ and $5 \in \mathcal{N}_3$. With $0 = (2 \times 5) - 7 - 3$, for $\mathcal{A}_{10-7}$, $(2 \times 1) = (2 \times 6) - 7 - 3$, for $\mathcal{A}_{12-7}$, $(2 \times 2) = (2 \times 7) - 7 - 3$, for $\mathcal{A}_{14-7}$, etc. Here $p^* = 7$ with $p = 3$.

Moreover, it is noted that the occurence of empty $\mathcal{A}_{...}$ sets is unavoidable. This is true for $2n > p^*$ cases. One can e.g. have $2n = 60$ and $p^* = 59$. The set $\mathcal{A}_{60-59} = \emptyset$, i.e. there are no $p \in \mathbb{P}$ such that, $2 < p \leq 1$. In addition we have empty $\mathcal{A}_{...}$ sets where $2n < p^* \Leftrightarrow 2n - p^* < 0$, i.e. there are no $p \in \mathbb{P}$ such that, $2 < p < 0$. Note that the upper bound, $2n - p^*$, of p in an $\mathcal{A}_{...}$ set is an $\mathbb{O}$ number.

Because $p^* + p$ is even, it is obvious that, $\mathcal{A}_{2n-p^*} \subseteq \mathbb{E}$. Hence, it can already be concluded that,

$$\mathcal{E} = \cup_{n \in \mathcal{N}_3} \cup_{p^* \in \mathbb{P}} \mathcal{A}_{2n-p^*} \subseteq \mathbb{E} \tag{5}$$

If there are non-Goldbach numbers, i.e. even numbers that cannot be composed out of the sum of two prime numbers, there is a non empty residue set for the generators

$$\mathcal{R} = \mathbb{E} \setminus \cup_{n \in \mathcal{N}_3} \cup_{p^* \in \mathbb{P}} \mathcal{A}_{2n-p^*} \tag{6}$$

This is so because then we can have (n, k) pairs where it is not possible to have $(p^*, p) \in \mathbb{P}^2$ with $2 < p \leq 2n - p^*$ and, for $n > k$, to have $2(n - k) = p^* + p$. Suppose, (one of) the number(s) is $2k \in \mathbb{E}$. Then, we have $2k \in \mathcal{R}$. This is $\Leftrightarrow$ with $2k \notin \cup_{n \in \mathcal{N}_3} \cup_{p^* \in \mathbb{P}} \mathcal{A}_{2n-p^*}$. For arbitrary (n, p^*) we, in a non trivial way, cannot find a $p \in \mathbb{P}$ with $2 < p \leq 2n - p^*$ such that $2k$ can be written with the use of n as $2n - p^* - p$. Or

$$2k \neq 2n - p^* - p \tag{7}$$

Of course there has to be a value attached to $2k \in \mathcal{R}$. This value can be considered yet unknown. For another value of n, another appropriate value of k applies but the impossibility $2k = 2n - p^* - p$ for any $(p^*, p) \in \mathbb{P}^2$ remains. From the examples one can see that there is an incidental way that $2k$ is not in $\mathcal{A}_{...}$. For instance $2k = 8 \notin \mathcal{A}_{46-23}$. However, finally we also have $2k = 8 \in \mathcal{A}_{60-23}$ or $8 \in \mathcal{A}_{46-19}$. Therefore $2k = 8 \in \mathcal{E}$. The required absence of any $(p^*, p) \in \mathbb{P}^2$ for $2k \in \mathcal{R}$ makes this impossibility not incidental.

According to the result of Pipping [5], the non-Goldbach is at its least $2(n-k) > 1 \times 10^5$. It is noted too that the argumentation below shows some traits of Lorenzen his proponent-opponent dialogic [7]. Here in (7), to have a $\mathcal{G}_\neg$ number, it is required not to have $2k$ *in any* $\mathcal{A}_{...}$.

If we note, $2k \notin \cup_{n\in\mathcal{N}_3} \cup_{p^*\in\mathbb{P}} \mathcal{A}_{2n-p^*}$, we are allowed to select a pair (n, p^*) with a set $\mathcal{A}_{2n-p^*} \neq \emptyset$. Because not any of the $\mathcal{A}_{...}$ sets may contain $2k$, an $\mathcal{A}_{...} \neq \emptyset$ set, with (n, p^*), can be selected such that there are predecessors of $2k$ that *do* reside in $\mathcal{A}_{2n-p^*} \neq \emptyset$. Hence, given k, we subsequently may have a finite $m \in \mathbb{N} > 0$, i.e. finite $m \in \mathcal{N}_0$ such that

$$2(k-m) = 2n - p^* - p > 0 \tag{8}$$

despite the truth of (7). Note that (7) should also apply for (n, p^*, p) used in (8). The $m \in \mathbb{N} > 0$, with $k > m$, is a genuine possibility because we already established that the potential $2k \in \mathbb{E}$ does not reside in $\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\} \subseteq \cup_{n\in\mathcal{N}_3} \cup_{p^*\in\mathbb{P}} \mathcal{A}_{2n-p^*}$. Suppose for illustration we are looking at e.g. $\mathcal{E} = \{0, 2, 4, \ldots, 20, \ldots 2(k-s)\}$. $2k \notin \mathcal{E}$ and s proper integer number.

To return to the main line of the argument, equation (8) implies that $2k = 2(n+m) - p^* - p$, or equivalently, $2k \in \mathcal{A}_{2(n+m)-p^*}$. Here $0 < k - m \leq k - s$. This set is, via selection of m, a "$\neq \emptyset$" set. Note now that $\mathcal{A}_{2(n+m)-p^*} \subseteq \cup_{n\in\mathcal{N}_3} \cup_{p^*\in\mathbb{P}} \mathcal{A}_{2n-p^*}$ with, of course, $(n+m) \in \mathcal{N}_3$. This runs contrary to the assumption that $2k \in \mathcal{R}$, or equivalently, $2k \notin \cup_{n\in\mathcal{N}_3} \cup_{p^*\in\mathbb{P}} \mathcal{A}_{2n-p^*}$.

Therefore, $\mathcal{R} = \emptyset$ and $\mathcal{E}$, defined in equation (4), is $\mathbb{E}$. For $0 < k - m \leq k - s$ we have found that $2k \in \mathcal{E}$. But we assumed $2k \in \mathcal{R} = \mathbb{E} \backslash \mathcal{E}$.

If one wants to have $2(n-k) \in \mathcal{G}_\neg$, then $2k$ must, exclusively, be in $\mathcal{R} = \mathbb{E} \backslash \mathcal{E}$ if n is allowed to vary along with k. So, $\mathcal{R}$ must be nonempty . However, $\mathcal{R}$ not empty is contradictory. Therefore $\mathcal{G}_\neg$ is empty.

DISCUSSION & CONCLUSION

When, $\cup_{n\in\mathcal{N}_3} \cup_{p^*\in\mathbb{P}} \mathcal{A}_{2n-p^*} = \mathbb{E}$, then for each $n \in \mathcal{N}_3 \subset \mathbb{N}$ there is a $k \in \mathbb{N}$, $k < n$, with $2k \in \cup_{n\in\mathcal{N}_3} \cup_{p^*\in\mathbb{P}} \mathcal{A}_{2n-p^*}$, such that we can always find a $p^* \in \mathbb{P}$ and a $p \in \mathbb{P}$. This $(p^*, p) \in \mathbb{P}^2$ pair has $2 < p \leq 2n - p^*$ such that $2(n-k) = p^* + p$.

If we represent the first sentences of this section in a formula, we have

$$\forall_{n\in\mathcal{N}_3} \forall_{k\in\mathbb{N}; n>k} \exists_{(p^*,p)\in\mathbb{P}^2;\ 2<p\leq 2n-p^*} \left[2(n-k) = p^* + p\right] \equiv true \tag{9}$$

The assumption of the existence of non-Goldbach even numbers

$$\exists_{n\in\mathcal{N}_3} \exists_{k\in\mathbb{N}} \nexists_{(p^*,p)\in\mathbb{P}^2;\ 2<p\leq 2n-p^*} \left[2(n-k) = p^* + p\right] \equiv true \tag{10}$$

gives a contradiction. Note that we are looking for one or more $2k$ that occur in *none* of the $\mathcal{A}_{2n-p^*}$. In that case the $\forall_{k\in\mathbb{N}}$ of (9) must be replaced by $\exists_{k\in\mathbb{N}}$. This replacement would allow (10). It was demonstrated that the assumption of a non-Goldbach number is contradictory.

Let us define the set $S \equiv \{(n, k) \,|\, n > k,\ n \in \mathcal{N}_3, k \in \mathbb{N}\}$. Hence, the set

$$U \equiv \{x \in \mathbb{N} \,|\, x = n - k,\ x > 3,\ (n, k) \in S\} \tag{11}$$

is $\sim$ to $\mathcal{N}_3$ [8].

Although, there are infinite many $(n, k) \in S$ associated to $x = n - k$ in (11), e.g. $x = 5 = 5 - 0 = 6 - 1 = 7 - 2 = \ldots$, the value of x occurs only once in U. Hence, $U \sim \mathcal{N}_3$.

Therefore, when firstly, $(n, k) \in S$ is connected to $(p^*, p) \in \mathbb{P}^2$, with $2 < p \leq 2n - p^*$, see (9). Secondly, the connection (n, k) and (p^*, p) is *not* 1-1. Thirdly there are no $(n, k) \in S$ not associated to some (p^*, p). In other words we only have incidental $2k \notin \mathcal{A}_{2n-p^*}$. The latter is true because it is *impossible* to find any k such that for each $n \in \mathcal{N}_3$ we are able to establish that $2k \notin \mathcal{A}_{2n-p^*}$ for all possible feasible $p^* \in \mathbb{P}$, i.e. those p^* for which $\mathcal{A}_{2n-p^*} \neq \emptyset$. The example is, for instance, that $(46, 8)$ is not based on $p^* = 23$, i.e. $8 \notin \mathcal{A}_{46-23}$. However, $(46, 8)$ is related to $p^* = 19$ and $p = 19$, i.e. $(46, 8)$ associated to $(19, 19)$, i.e. $2 < p \leq 46 - 19 = 27$, such that $8 \in \mathcal{A}_{46-19}$.

It then follows from the defnitions of S and U sets that, all $\alpha \in 2\mathcal{N}_3$ are represented in $2U$ [9]. Each $2(n-k)$ in $2U$ is, therefore, associated to a pair (p^*, p). Hence, referring to equation (9), each $\alpha \in \mathbb{E}$ can, at its least, be written as the sum of some particular $p^* \in \mathbb{P}$ and some particular $p \in \mathbb{P}$, such as represented in (9).

We have demonstrated that a set of possible non-Goldbach even numbers must be empty, $\mathcal{G}_\neg = \emptyset$. This means there are no non-Goldbach even numbers. We have discussed why the employed criterium for generators is correct. Therefore the conjecture of Goldbach is true.

REFERENCES

[1] A.V. Kumchev & D.I. Tolev, *An invitation to additive prime number theory*, arxiv:math/0412220v1, (2004).

[2] I.M. Vinogradov, *On the representation of an odd number as a sum of three primes*, Translation in *Goldbach Conjecture* ed. Wang Yuan, World Scientific (1984).

[3] J.G. van der Corput, *Sur l'hypothese de Goldbach*, Proc. Akad. Wet. Amsterdam 41, 76-80, (1938).

[4] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen I*, Druck & Verlag B.G. Teubner, Leipzig, (1909).

[5] N. Pipping, *Die Goldbachsche Vermutung und der Goldbach Vinogradov Satz*, Acta Acad. Aboenis, Math. Phys. 11, 4-25, (1938).

[6] S. Marshall, *Elementary proof of the Goldbach conjecture*, vixra: 1702.0150v1.

[7] P. Lorenzen, *Einfürung in die operatieve Logik und Mathematik*, Berlin Springer (1955).

[8] A. Hajnal & P. Hamburger, *Set Theory*, Lond. Math. Soc. Student Text, 48, page 11, (1999).

[9] V. Lakshmikantham, T. Gnana Bashkar & J. Vasundhara Devi, *Theory of Set Differential Equations in Metric Spaces*, Florida Institute of Technology, page 6, (2004).