Question 474: A DEFINITE INTEGRAL

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abstract
This note presents a definite integral

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En una nota anterior revisamos integrales del tipo:

$$\int_0^{\pi} x\sqrt{a+b\cos^2 x+c\sqrt{d+e\cos^2 x+f\cos^4 x}}\,\sin x\,dx \tag{1}$$

En esta nota revisamos el caso particular:

$$a=2, b=\sqrt{3}, c=2, d=1, e=\sqrt{3}, f=1 \tag{2}$$

La Integral

$$I=\int_0^{\pi} x\sqrt{2+\sqrt{3}\cos^2 x+2\sqrt{1+\sqrt{3}\cos^2 x+\cos^4 x}}\,\sin x\,dx=$$

$$=\frac{\pi}{4\sqrt{2}}\left\{2\sqrt{2\left(2+\sqrt{2}+\sqrt{3}+\sqrt{6}\right)}+2\left(-1+\sqrt{3}\right)\tan^{-1}\left(\sqrt{-1-\sqrt{2}+\sqrt{6}}\right)+\right. \tag{3}$$

$$\left.+\left(1+\sqrt{3}\right)\ln\left(1+\frac{\sqrt{2}+\sqrt{6}+2\sqrt{2+\sqrt{2}+\sqrt{3}+\sqrt{6}}}{2}\right)\right\}$$

Algunas relaciones:

$$I=\int_{-1}^{1}\sqrt{2+\sqrt{3}x^2+2\sqrt{1+\sqrt{3}x^2+x^4}}\,\cos^{-1}x\,dx \tag{4}$$

$$I=\pi\int_0^{1}\sqrt{2+\sqrt{3}x^2+2\sqrt{1+\sqrt{3}x^2+x^4}}\,dx \tag{5}$$

$$I=\pi\int_0^{\pi/2}\sqrt{2+\sqrt{3}\sin^2 x+2\sqrt{1+\sqrt{3}\sin^2 x+\sin^4 x}}\,\cos x\,dx \tag{6}$$

$$I=\pi\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}-\pi\int_2^{\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}\sqrt{2x\sqrt{x^2-1}-\sqrt{3}x^2}\,dx \tag{7}$$

$$I=\frac{\pi\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}{2}+2\pi\int_{\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}^{\infty}\frac{x^2}{x^4-2\sqrt{3}\,x^2-1}\,dx \tag{8}$$

$$I=\frac{\pi\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}{2}+2\pi\int_{0}^{1/\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}\frac{1}{1-2\sqrt{3}\,x^2-x^4}\,dx \tag{9}$$

$$I=\pi\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}-\frac{\pi}{2}\int_{4}^{2+\sqrt{2}+\sqrt{3}+\sqrt{6}}\sqrt{\frac{2\sqrt{x^2-x}-\sqrt{3}\,x}{x}}\,dx \tag{10}$$

$$I=2\pi\,\mathrm{Re}\left\{{}_2F_1\left(\frac{1}{2},-\frac{1}{2};\frac{3}{2};-\frac{\sqrt{3}+i}{2}\right)\right\} \tag{11}$$

$$I=2\pi\,\mathrm{Re}\left\{\sqrt{\frac{2}{2+\sqrt{3}+i}}\;{}_2F_1\left(\frac{1}{2},2;\frac{3}{2};\frac{1+i\left(2-\sqrt{3}\right)}{2}\right)\right\} \tag{12}$$

$$I=2\pi\,\mathrm{Re}\left\{\sqrt{\frac{2+\sqrt{3}+i}{2}}\;{}_2F_1\left(-\frac{1}{2},1;\frac{3}{2};\frac{1+i\left(2-\sqrt{3}\right)}{2}\right)\right\} \tag{13}$$

$$I=2\pi\,\mathrm{Re}\left\{\left(\frac{2+\sqrt{3}+i}{2}\right)^{3/2}\;{}_2F_1\left(1,2;\frac{3}{2};-\frac{\sqrt{3}+i}{2}\right)\right\} \tag{14}$$

$$I=\frac{\pi\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}{2}+ \\ +\frac{2\pi}{\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}\sum_{n=0}^{\infty}\left(\frac{2\sqrt{3}}{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}\right)^{n}\sum_{k=0}^{n}\binom{n}{k}\frac{\left(6+4\sqrt{3}+4\sqrt{6+3\sqrt{3}}\right)^{-k}}{2n+2k+1} \tag{15}$$

$$I=\frac{\pi\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}{2}+ \\ +\frac{2\pi}{\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}\sum_{n=0}^{\infty}\left(\frac{2\sqrt{3}}{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}\right)^{n}\frac{1}{2n+1}\sum_{k=0}^{[n/2]}\binom{n-k}{k}12^{-k} \tag{16}$$

$$I=\frac{\pi\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}{2}+$$
$$+\frac{\pi}{2\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}\sum_{n=0}^{\infty}\frac{\left(2+\sqrt{3}+2\sqrt{2+\sqrt{3}}\right)^{-n}}{2n+1}\left(\left(2+\sqrt{3}\right)^{n+1}+\left(-1\right)^{n}\left(2-\sqrt{3}\right)^{n+1}\right) \tag{17}$$

$$I=\frac{\pi\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}{2}+$$
$$+\frac{\pi}{2\sqrt{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}}\left(\left(2-\sqrt{3}\right){}_2F_1\left(1,\frac{1}{2};\frac{3}{2};-\frac{2-\sqrt{3}}{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}\right)+\right. \tag{18}$$
$$\left.+\left(2+\sqrt{3}\right){}_2F_1\left(1,\frac{1}{2};\frac{3}{2};\frac{2+\sqrt{3}}{2+\sqrt{3}+2\sqrt{2+\sqrt{3}}}\right)\right)$$

Observaciones:

- ${}_2F_1\left(a,b;c;x\right)=\sum_{n=0}^{\infty}\frac{\left(a\right)_n\left(b\right)_n}{\left(c_n\right)}\frac{x^n}{n!}$, es la clásica función hipergeométrica.
- $\left(a\right)_n=a\left(a+1\right)...\left(a+n-1\right)\quad,\left(a\right)_0=1$
- $i=\sqrt{-1}\quad,\mathrm{Re}\left(z\right)$ es la parte real z .
- $2+\sqrt{3}+2\sqrt{2+\sqrt{3}}=2+\sqrt{2}+\sqrt{3}+\sqrt{6}$

Referencias

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