On a singular Kamke equation equivalent to linear harmonic oscillator

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Abstract

This paper investigates analytical properties of a singular Kamke second order equation consisting of a generalization of Kamke equation 6.110 and Kamke equation 6.111. It is shown that this equation belongs to the class of quadratic Liénard type equations closely related to the linear harmonic oscillator and introduced recently by some authors of this work. In this way, exact Kamke solutions are recovered. The connection between this equations and the linear harmonic oscillator shows its possible use in mechanical vibrations study and quantum mechanics.

Introduction

In his book [1], Kamke considered the singular second order equation 6.111 of quadratic Liénard type

$$\ddot{x}(t)-\frac{\dot{x}(t)^2}{x(t)}-\frac{1}{x(t)}=0 \tag{1}$$

with the exact solution given in the form [1]

$$C_1 x(t)=C_0(C_1 t+C_2) \tag{2}$$

He considered also the equation 6.110 [1]

$$\ddot{x}(t)-\frac{\dot{x}(t)^2}{x(t)}+\frac{1}{x(t)}=0 \tag{3}$$

with solution

$$x(t)=C_1\sin(C_1 t+C_2) \tag{4}$$

These equations may be considered as particular cases of reduced form of Kamke equation 6.165.

Generalized singular equation

To study the generalized form of these equations, it appears intersting to consider the theory of nonlinear differential equations recently introduced by Akande et al. [2]. In this regard one may take into account the general class of quadratic Liénard equations [2]

$$\ddot{x}(t)+(l-2\gamma)\frac{\dot{x}(t)^2}{x(t)}+\frac{a^2}{l+1}x(t)^{4\gamma+1}=0 \tag{5}$$

where l, γ and a are free paramerters. For $l-2\gamma=-1$, equation (5) reduces to

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$$\ddot{x}(t)-\frac{\dot{x}(t)^2}{x(t)}+\frac{a^2}{2\gamma}x(t)^{4\gamma+1}=0 \qquad (6)$$

By application of $\gamma=-\frac{1}{2}$, equation (6) becomes

$$\ddot{x}(t)-\frac{\dot{x}(t)^2}{x(t)}-\frac{a^2}{x(t)}=0 \qquad (7)$$

which may give equation (1) for $a=1$, and equation (3) for $a^2=-1$. Clearly in this way equation (7) denotes a generalized form of (1) and (3) but a reduced form of Kamke equation 6.165. According to the theory of Akande et al. [2], the exact and explicit general solution to the generalized singular differential equation (7) may read

$$x(t)=-\frac{1}{A_0}\frac{1}{\sin(a\phi(t)+\alpha)} \qquad (8)$$

where A_0, a and α are arbitrary parameters such that

$$-A_0(t+K)=\int\frac{d\phi}{\sin(a\phi+\alpha)} \qquad (9)$$

and K denotes a constant of integration. The evaluation of integral of right hand side leads to

$$-A_0(t+K)=\frac{1}{a}\ln\left[\tan(\frac{\theta}{2})\right] \qquad (10)$$

where $\theta=a\phi+\alpha$. From (10) one may obtain after a little mathematical treatment

$$x(t)=-\frac{1}{A_0}\frac{1}{\sin\left[2tg^{-1}(\exp(-aA_0(t+K)))\right]} \qquad (11)$$

Non-periodic solution

The use of trigonometric identity

$$\sin\left[2tg^{-1}(\exp(-aA_0(t+K)))\right]=\frac{1}{\cosh\left[-aA_0(t+K)\right]} \qquad (12)$$

leads to

$$x(t)=-\frac{1}{A_0}\cosh\left[aA_0(t+K)\right] \qquad (13)$$

Equation (13) designates the exact and explicit general solution to the generalized equation (7). Thus, when $a^2 \geq 0$, for $a = 1$, equation (13) gives an exact and explicit solution to the Kamke equation 6.111 in the form

$$x(t) = -\frac{1}{A_0}\cosh\left[A_0(t+K)\right] \tag{14}$$

Trigonometric periodic solution

For $a = ib$, where i is the purely imaginary number and b an arbitrary parameter, when $a^2 \prec 0$, the exact and explicit general solution (13) takes the trigonometric expression

$$x(t) = -\frac{1}{A_0}\cos\left[bA_0(t+K)\right]$$

that is

$$x(t) = \frac{1}{A_0}\cos\left[bA_0 t + (bA_0 K + \pi)\right] \tag{15}$$

Putting $C_1 = A_0$, and $C_2 - \frac{\pi}{2} = bA_0 K + \pi$, the exact and general trigonometric solution to the generalized singular equation (7) may be writting in the definitive form

$$x(t) = \frac{1}{C_1}\sin(bC_1 t + C_2) \tag{16}$$

For $b = 1$, one may recover the explicit solution (4) given by Kamke to equation (3) in his book [1]. The exact and general trigonometric periodic solution (16) is of harmonic form but with amplitude dependent frequency. Therefore as mentioned in [2] there are several classes of quadratic Liénard type equations which may exhibit exact and explicit solutions of harmonic form. Such a finding confirms the theory of nonlinear differential equation introduced recently by Akande et al.[2].

References

[1] E. Kamke, Differentialgleichungen lösungsmethoden und lösungen, Springer Fachmedien Weisbaden GMBH, 10th edition, 1977.

[2] J. Akande, D. K. K. Adjaï, L. H. Koudahoun, Y. J. F. Kpomahou, M. D. Monsia, Theory of exact trigonometric periodic solutions to quadratic Liénard type equations, viXra: 1704.0199 v1 (2017).