Navier-Stokes Three Dimensional Equations General Periodic Solutions for a given Periodic Initial Velocity Vector Field for Incompressible Fluid with Positive Viscosity

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Abstract

The existence of smooth periodic solutions for Navier-Stokes three dimensional equations for a given periodic initial velocity vector field with positive viscosity is proved. The equation is solved by considering Fourier series representation of periodic initial velocity vector fields and predicting the velocity vector field at all times. The solution discovered here can also be used as counter example for clay mathematics millennium prize problem.

1. Introduction

Navier-Stokes equation is partial differential equation relating velocity vector field, scalar pressure field and external force vector field acting on fluid having viscosity. The equation is derived in 1845 by C.Navier and G.Stoke. The solution for three dimensional version of the equation resisted mathematician for centuries. Due to its great importance for the theory of differential equations and fluid dynamics, the proof for existence and smoothness of solutions of Navier-Stokes equations is clay mathematics institute millennium prize problem.

The solution is derived by representing periodic initial velocity vector field as the sum of sine and cosine series with properly chosen argument and coefficients. This paper describes all conditions required for the solution to be physically reasonable as posted by clay mathematics institute, drive solutions for the problem and prove results fulfill all conditions described.

1.1 Statement of the problem

For a given a, b, c elements of real number and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be unit vectors in the direction x, y, z respectively.

$\vec{U}$: is velocity vector field on Euclidean space depending on space coordinates and varying in time.

p: is scalar pressure field function of also four variables.

$\vec{F}$: External force field acting on the fluid in which on this paper set to be zero to follow clay mathematics conditions.

$\vec{\mathrm{U}}^0$: Initial velocity vector field at time zero function of only three space variables.

$$\vec{\nabla} g(x, y, z) = \mathbf{i}\mathrm{a} + \mathbf{j}\mathrm{b} + \mathbf{k}\mathrm{c} \tag{1.1.0}$$

$$\vec{\mathrm{R}} = \mathbf{i}\mathrm{x} + \mathbf{j}\mathrm{y} + \mathbf{k}\mathrm{z} \tag{1.1.1}$$

Navier-Stokes equations are shown below.

$$\frac{\partial \vec{U}}{\partial t} + \left(\vec{U} \cdot \vec{\nabla}\right)\vec{U} = \nu \nabla^2 \vec{U} - \vec{\nabla}\mathrm{p} + \vec{\mathrm{F}} \tag{1.1.2}$$

$$\frac{\partial \vec{U}}{\partial t} + \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right)\vec{U} = \nu \nabla^2 \vec{U} - \vec{\nabla}\mathrm{p} + \vec{\mathrm{F}} \tag{1.1.3}$$

Divergence free velocity vector field.

$$\vec{\nabla} \cdot \vec{\mathrm{U}} = 0 \tag{1.1.4}$$

Initial condition, velocity vector field at time zero is given.

$$\vec{U}|_{t=0} = \vec{U}^0 \tag{1.1.5}$$

Periodic Nature of initial velocity vector field is showed.

$$\vec{U}^0(x, y, z) = \vec{U}^0(x + a , y + b , z + c) \tag{1.1.6}$$

Periodic nature of Velocity vector field at all times.

$$\vec{U}\,(x, y, z, t) = \vec{U}\,(x + a , y + b , z + c, t) \tag{1.1.7}$$

External force is periodic with period vector $\vec{A}$.

$$\vec{F}(x, y, z, t) = \quad \vec{F}(x + a, y + b, z + c, t) \tag{1.1.8}$$

Velocity vector field and scalar pressure are defined for all real positions and positive time.

$$\vec{U}\,, p \ \in C^{\infty}\big(R^n \times [0, \infty)\big) \tag{1.1.9}$$

1.2 Initial condition

Let $g(x, y, z)$ be a scalar function of three variables x, y, z having gradient of constant vector which is the same as period vector of initial velocity vector field given. For any position vector $\vec{R}$, the function $g(x, y, z)$ can be written as follows.

$$g(x, y, z) = \vec{R} \cdot \vec{\nabla} g(x, y, z)$$

Any periodic vector field having period vector of $\vec{\nabla} g(x, y, z)$ can be written as infinite sum of sine and cosine series by properly choosing the argument and coefficients. The vector field can be represented as Fourier series by considering the whole function $g(x, y, z)$ as one variable. Due to linearity of vectors all components can be represented as Fourier series sum.

$$\vec{U}^0\big(g(x, y, z)\big) = \vec{U}^0\left(\vec{R} \cdot \vec{\nabla} g(x, y, z)\right)$$

For the function is periodic with period vector $\vec{\nabla} g(x, y, z)$ replace $\vec{R}$ with $\vec{R} + \vec{\nabla} g(x, y, z)$

$$\vec{U}^0\left(\vec{R} \cdot \vec{\nabla} g(x, y, z)\right) = \vec{U}^0\left((\vec{R} + \vec{\nabla} g(x, y, z)) \cdot \vec{\nabla} g(x, y, z)\right)$$

$$\vec{U}^0\left(\vec{R} \cdot \vec{\nabla} g(x, y, z)\right) = \vec{U}^0\big((\vec{R} \cdot \vec{\nabla} g(x, y, z)) + (\vec{\nabla} g(x, y, z) \cdot \vec{\nabla} g(x, y, z))\big)$$

To simplify the problem we get rid of the variables x, y, z and work with one variable g only.

$$\vec{U}^0\big(\vec{R} \cdot \vec{\nabla} g\big) = \vec{U}^0\big((\vec{R} \cdot \vec{\nabla} g) + (\vec{\nabla} g \cdot \vec{\nabla} g)\big)$$

$$\vec{U}^0(g) = \vec{U}^0\big(g + (\vec{\nabla} g \cdot \vec{\nabla} g)\big)$$

From the above relation we can state our periodic vector field is one variable and periodic with a period of

$$\vec{\nabla} g \cdot \vec{\nabla} g = |\vec{\nabla} g|^2$$

Therefore we can represent any periodic vector field with Fourier series by properly choosing the coefficients and arguments.

$$\vec{\mathrm{U}}^{0}(x,y,z)=\vec{\mathrm{a}}_0+\sum_{n=1}^{\infty}\left(\vec{\mathrm{a}}_{\mathrm{n}}\cos\left(2n\pi\, g\right)+\vec{\mathrm{b}}_{\mathrm{n}}\sin\left(2n\pi\, g\right)\right) \tag{1.2.0}$$

$$\vec{\mathrm{a}}_0=\frac{1}{2}\frac{1}{|\vec{\nabla}g|^2}\int_{-|\vec{\nabla}g|^2}^{|\vec{\nabla}g|^2}\vec{\mathrm{U}}^{0}(g)\,dg$$

$$\vec{\mathrm{a}}_{\mathrm{n}}=\frac{1}{|\vec{\nabla}g|^2}\int_{-|\vec{\nabla}g|^2}^{|\vec{\nabla}g|^2}\vec{\mathrm{U}}^{0}(g)\cos\left(2n\pi\, g\right)dg$$

$$\vec{\mathrm{b}}_{\mathrm{n}}=\frac{1}{|\vec{\nabla}g|^2}\int_{-|\vec{\nabla}g|^2}^{|\vec{\nabla}g|^2}\vec{\mathrm{U}}^{0}(g)\sin\left(2n\pi\, g\right)dg$$

Initial velocity vector field is divergence free

$$\vec{\nabla}\cdot\vec{\mathrm{U}}^{0}(g)=\vec{\nabla}g\cdot\frac{\mathrm{d}}{\mathrm{dg}}\vec{\mathrm{U}}^{0}(g)$$

Because $\vec{\nabla}g$ is constant vector the above equation can be rewritten as

$$\vec{\nabla}\cdot\vec{\mathrm{U}}^{0}(g)=\frac{\mathrm{d}}{\mathrm{dg}}(\vec{\nabla}g\cdot\vec{\mathrm{U}}^{0}(g))$$

We can use now the divergence free vector field idea

$$0=\frac{\mathrm{d}}{\mathrm{dg}}(\vec{\nabla}g\cdot\vec{\mathrm{U}}^{0}(g))$$

This shows the dot product $\vec{\nabla}g\cdot\vec{\mathrm{U}}^{0}(g)$ is constant number.

Dot product of the equation (1.2.0) with $\vec{\nabla}g$ results only one constant number $\vec{\nabla}g\cdot\vec{\mathrm{a}}_0$ the other is variable function. Since the dot product $\vec{\nabla}g\cdot\vec{\mathrm{U}}^{0}(g)$ is constant the dot product of variable term with $\vec{\nabla}g$ has to vanish.

$$\vec{\nabla}g\cdot\vec{\mathrm{U}}^{0}(g)=\vec{\mathrm{a}}_0\cdot\vec{\nabla}g$$

And

$$0=\sum_{n=1}^{\infty}\left(\vec{\mathrm{a}}_{\mathrm{n}}\cdot\vec{\nabla}g\cos\left(2n\pi\, g\right)+\vec{\mathrm{b}}_{\mathrm{n}}\cdot\vec{\nabla}g\sin\left(2n\pi\, g\right)\right)$$

2. Solution of Navier-Stokes equation in the absence of external force.

From the initial velocity vector field we predict the velocity vector field at any later time.

Let

$$l(x,y,z,t)=g(x,y,z)+k(t)$$

To avoid bulky equations we will use l in place of $l(x,y,z,t)$

$$\vec{\mathrm{U}}\,(l) = \vec{\mathrm{a}}_0 h(t) + \sum_{n=1}^{\infty} \left(\vec{\mathrm{a}}_\mathrm{n} \cos\,(2n\pi\, l) + \vec{\mathrm{b}}_\mathrm{n} \sin\,(2n\pi\, l)\right)\mathrm{H(n,t)} \tag{2.1}$$

Split Navier-Stokes equations in to different parts and substitute the above solution (2.1) to each.

First time derivative of the velocity vector $\frac{\partial \vec{U}}{\partial t}$, is simplified to the following expression.

$$\frac{\partial \vec{U}}{\partial t} = \vec{\mathrm{a}}_0 \frac{\partial}{\partial t} h(t) + \sum_{n=1}^{\infty} \left(\vec{\mathrm{a}}_\mathrm{n} \cos(2n\pi\, l) + \vec{\mathrm{b}}_\mathrm{n} \sin(2n\pi\, l)\right) \frac{\mathrm{d}}{\mathrm{dt}} \mathrm{H(n,t)} + \sum_{n=1}^{\infty} 2n\pi\left(-\vec{\mathrm{a}}_\mathrm{n} \sin(2n\pi\, l) + \vec{\mathrm{b}}_\mathrm{n} \cos(2n\pi\, l)\right)\mathrm{H(n,t)} \frac{\mathrm{d}}{\mathrm{dt}} \mathrm{k(t)}$$

The next term$\left(\vec{U} \cdot \vec{\nabla}\right)\vec{U}$, is simplified to the following equation.

$$\left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) \vec{U} = \vec{U} \cdot \vec{\nabla} g \sum_{n=1}^{\infty} 2n\pi\left(-\vec{\mathrm{a}}_\mathrm{n} \sin(2n\pi\, l) + \vec{\mathrm{b}}_\mathrm{n} \cos(2n\pi\, l)\right)\mathrm{H(n,t)} \tag{2.3}$$

$$\left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) \vec{U} = \sum_{n=1}^{\infty} 2n\pi\left(-\vec{\mathrm{a}}_\mathrm{n} \sin(2n\pi\, l) + \vec{\mathrm{b}}_\mathrm{n} \cos(2n\pi\, l)\right) h(t) \vec{A} \cdot \vec{\mathrm{a}}_0 \mathrm{H(n,t)} \tag{2.4}$$

The final term $\nabla^2 \vec{U}$, is simplified to the following expression.

$$\nu \nabla^2 \vec{U} = -\sum_{n=1}^{\infty} \left(\vec{\mathrm{a}}_\mathrm{n} \cos(2n\pi\, l) + \vec{\mathrm{b}}_\mathrm{n} \sin(2n\pi\, l)\right) \nu (2n\pi)^2 |\vec{\nabla} g|^2 \mathrm{H(n,t)} \tag{2.5}$$

Collecting all results of simplification we get the following equation.

$$-\vec{\nabla}\mathrm{p} = \vec{\mathrm{a}}_0 \frac{\partial}{\partial t} h(t) + \sum_{n=1}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{dt}} \mathrm{H(n,t)} + \nu(2n\pi)^2 |\vec{\nabla} g|^2 \mathrm{H(n,t)}\right) \left(\left(\vec{\mathrm{a}}_\mathrm{n} \cos(2n\pi\, l) + \vec{\mathrm{b}}_\mathrm{n} \sin(2n\pi\, l)\right)\right) + \sum_{n=1}^{\infty} \left(\vec{\nabla} g \cdot \vec{\mathrm{a}}_0 h(t) + \frac{\mathrm{dk(t)}}{\mathrm{dt}}\right) \mathrm{H(n,t)} \left(2n\pi\left(-\vec{\mathrm{a}}_\mathrm{n} \sin(2n\pi\, l) + \vec{\mathrm{b}}_\mathrm{n} \cos(2n\pi\, l)\right)\right) \tag{2.6}$$

One of many possible solutions is where the scalar pressure is non-periodic. As a result the component with oscillating term shown above in the equation (2.6) has to vanish.

$$\left(\frac{\mathrm{d}}{\mathrm{dt}} \mathrm{H(n,t)} + \nu(2n\pi)^2 |\vec{\nabla} g|^2 \mathrm{H(n,t)}\right) = 0 \tag{2.7}$$

$$\left(\vec{\nabla} g \cdot \vec{\mathrm{a}}_0 h(t) + \frac{\mathrm{dk(t)}}{\mathrm{dt}}\right) = 0 \tag{2.8}$$

Solving equation (2.7) the following is the solution

$$\mathrm{H(n,t)} = \mathrm{e}^{-\upsilon |\vec{\nabla} g|^2 (2n\pi)^2 \mathrm{t}}$$

Similarly solve equation (2.8) results the following

$$\mathrm{k(t)} = -\int_0^t \vec{\nabla} g \cdot \vec{\mathrm{a}}_0 h(\tau) d\tau$$

3. Summary of result

Therefore smooth periodic velocity vector field and scalar pressure field solutions are

$$\vec{\mathrm{U}}\,(x,y,z,t) = \vec{\mathrm{a}}_0 h(t) + \sum_{n=1}^{\infty} \left(\vec{\mathrm{a}}_\mathrm{n} \cos\,(2n\pi\, l(x,y,z,t) + \vec{\mathrm{b}}_\mathrm{n} \sin\,(2n\pi\, l(x,y,z,t)) \right) \mathrm{H(n,t)}$$

$$-p(x,y,z,t) = \vec{\mathrm{a}}_0 \cdot \vec{R}\, \frac{\partial}{\partial t} h(t)$$

Where

$$\vec{\mathrm{a}}_0 = \frac{1}{2} \frac{1}{|\vec{\nabla} g|^2} \int_{-|\vec{\nabla} g|^2}^{|\vec{\nabla} g|^2} \vec{\mathrm{U}}^0(g)\, dg$$

$$\vec{\mathrm{a}}_\mathrm{n} = \frac{1}{|\vec{\nabla} g|^2} \int_{-|\vec{\nabla} g|^2}^{|\vec{\nabla} g|^2} \vec{\mathrm{U}}^0(g) \cos\,(2n\pi\, g)\, dg$$

$$\vec{\mathrm{b}}_\mathrm{n} = \frac{1}{|\vec{\nabla} g|^2} \int_{-|\vec{\nabla} g|^2}^{|\vec{\nabla} g|^2} \vec{\mathrm{U}}^0(g) \sin\,(2n\pi\, g)\, dg$$

$$l(x,y,z,t) = g(x,y,z) - \vec{\nabla} g \cdot \vec{\mathrm{a}}_0 \int_0^t h(\tau) d\tau$$

$$\mathrm{H(n,t)} = \mathrm{e}^{-\upsilon |\vec{\nabla} g|^2 (2n\pi)^2 \mathrm{t}}$$

$$\lim_{\mathrm{t}\to\infty} |\mathrm{h(t)}| < \infty\,, h(0) = 1$$

4. Conclusions

1. Any periodic vector field can be represented as the sum of sine and cosine series with proper vector constant coefficients and properly chosen arguments.
2. For any periodic initial velocity vector field represented as the sum of sine and cosine series there exist solutions for Navier-Stokes equations as shown above.

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