Perfect Cuboid

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Abstract:There is no cuboid with all integer edges and face diagonals.

Problem: Euler Box is a cuboid/rectangular box in which all edges and all face diagonals are integers. The Perfect Cuboid would be an Euler Box with the integer space diagonals. PROVE OR DISPROVE THAT THERE IS A PERFECT CUBOID.

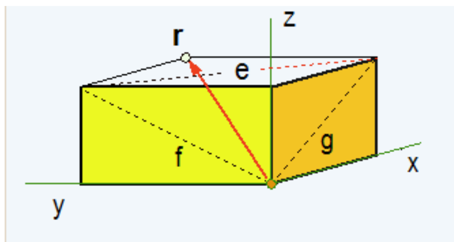

Figure 1: Euler Box

In the Euler Box of the edges (x, y, z) placed in the coordinate origin on the Figure 1, the following relations between its edges and all diagonals hold

$$x^2 + y^2 = e^2, \quad y^2 + z^2 = f^2, \quad z^2 + x^2 = g^2,$$
$$x^2 + y^2 + z^2 = r^2, \quad e^2 + f^2 + g^2 = 2r^2.$$

Clearly, the same relations hold among the relative segments $\bar{x} = x/r, \bar{y} = y/r, \bar{z} = z/r$ and $\bar{e} = e/r, \bar{f} = f/r$ and $\bar{g} = g/r$, so that an equivalent problem is TO FIND IF THERE IS A PERFECT CUBOID IN THE RATIONAL NUMBERS.

Corollary 01. The Perfect Cuboid Problems in Cartesian and Spherical coordinates are equivalent.

■ In the spherical coordinate system $(r; \theta, \phi)$ the Euler Box is completely defined by

$$x = r \sin\theta \sin\phi, \quad y = r \sin\theta \cos\phi. \quad z = r \cos\theta,$$

The Euler Box diagonals are

$$\begin{aligned}
e &= \sqrt{x^2 + y^2} = r \sin\theta, \\
f^2 &= y^2 + z^2 = r^2(1 - \sin^2\theta \sin^2\phi), \\
g^2 &= z^2 + x^2 = r^2(1 - \sin^2\theta \cos^2\phi), \\
d &= \sqrt{x^2 + y^2 + z^2} = r.
\end{aligned}$$

To prove that an Euler Box is the Perfect Cuboid is the same as to prove that all segments $\{x, y, z;\ e, f, g, d\}$ are the rational numbers in the variables $(r; \theta, \phi)$. $\square$

Definition: *The collection of all couples* $(u, v) = (\sin\varpi, \cos\varpi)$ *with norm* $|u, v| = u^2 + v^2 = 1$ *are the trigonometric or unit number pairs. Such are* $(s, c) = (\sin\theta, \cos\theta)$ *and* $(\sigma, \omega) = (\sin\phi, \cos\phi)$.
The collection of all integer triples $(\alpha, \beta; \gamma) = 1$, $\alpha^2 + \beta^2 = \gamma^2$, *are the Pythagorean minimal triples, and the collection of all rational pairs* $(\alpha, \beta) \sim (\alpha, \beta; 1)$ *are the trigonometric Pythagorean pairs.*
Two non-rational numbers are the multiplication rational if their product is a rational number; examples are $7\sqrt{3} \cdot \sqrt{3}/6 = 7/2$, *and* $\ln\pi \cdot 3/\ln\pi^4 = 3/4$. *A non-rational number is the square rational if its square is a rational number; for example such is* $\sqrt{2}$.

Further we will use the unit number pairs $(s, c) = (\sin\theta, \cos\theta)$ and $(\sigma, \omega) = (\sin\phi, \cos\phi)$, and the variable pair Φ, Ψ and rewrite

$$\begin{aligned}
&x = rs\sigma \quad y = rs\omega. \quad z = rc,\\
&e\ = rs, \qquad\ d = r.\\
&f^2 = \Phi r^2 : \quad \Phi = 1 - s^2\sigma^2,\\
&g^2 = \Psi r^2 : \quad \Psi = 1 - s^2\omega^2.
\end{aligned}$$

Corollary 02. For an Euler Box to be Perfect Cuboid it is necessary that $\sqrt{\Phi}$ and $\sqrt{\Psi}$ are rational.

■ The conclusion follows from the rational number nature of the edges and face diagonals of the Euler Box. The segments z and e require r and (s, c) to be either both rational or multiplication rational. In either case, since the segments x and y are rational, the pair (σ, ω) must be strictly rational.
Further, irrespective of the structure of the products $r \cdot (\mathrm{s}, \mathrm{c})$ the rationality of the diagonals f and g forces the pair $(\sqrt{\Phi}, \sqrt{\Psi})$ to be rational. For, if $(\sqrt{\Phi}, \sqrt{\Psi})$ are not rational the r must be non-rational and the Euler Box is not the Perfect Cuboid. We accounted for the rationality of all segments without specifying the number nature of space diagonal r. *Hence, the rationality of the pair* $(\sqrt{\Phi}, \sqrt{\Psi})$ *is the necessary condition for an Euler Box to be the Perfect Cuboid.* $\square$

Remark: The Perfect Cuboid's necessary condition is equivalent to the requirement that the functions Φ and Ψ are the perfect rational squares. Thus must be

$$1 - \mathrm{s}^2\sigma^2 = \xi^2, \qquad 1 - \mathrm{s}^2\omega^2 = \eta^2 \tag{1}$$

$$\Updownarrow$$

$$\eta^2 + \xi^2 = 2 - \mathrm{s}^2, \quad \eta^2 - \xi^2 = \mathrm{s}^2(\sigma^2 - \omega^2). \tag{2}$$

The last equation in its factored form $(\eta + \xi)(\eta - \xi) = (\mathrm{s}\sigma)^2 - (\mathrm{s}\omega)^2$ has a useful geometric interpretation, which we will give after the following observation.

Remark: The right triangle $\triangle = (u, v; w)$ on the Figure 2.1 of the hypotenuse w and the segments p and q cut on the hypotenuse by its hight h, $h^2 = pq$, inscribed in the circle is unique. The hypotenuse hight partitions the triangle into two unique, right, mutually complementary triangles $(p, h; a)$ and $(q, h; b)$, $(u, v; w) = (p, h; u) \oplus (q, h; v)$.

Other way around, the triangle $(p, h; a)$ on the Figure 2.2 defines unique segment p on a ray x, and the segments q and b by the ray orthogonal to its hypotenuse a. Created right triangles $(q, h; b)$ and $(a, b; w)$; are unique and $w = p + q$, $(a, b; w) = (p, h; a) \oplus (q, h; b)$.

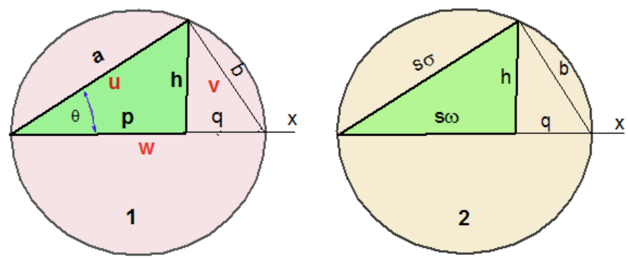

Figure 2: Euler Box Geometry

Consequently the following two equations are naturally associated with the right triangle $(p, h; a)$

$$a^2 - p^2 = h^2 = pq, \qquad b^2 - q^2 = h^2 = pq.$$

The product pq may not be the perfect square

Corollary 03. If an Euler Box is the Perfect Cuboid the numbers ξ and η must be solutions to the quadratic equation

$$\begin{aligned} & \mathrm{Z}^2 - \mathrm{s}\omega \mathrm{Z} + 2^{-1}[\mathrm{s}^2(1+\omega^2) - 2\,] = 0, \\ \text{AND}: \quad & \eta = \ \ 2^{-1}(\mathrm{s}\omega + \sqrt{2+\mathrm{s}^2}), \\ & \xi = \pm 2^{-1}(\mathrm{s}\omega - \sqrt{2+\mathrm{s}^2}). \end{aligned}$$

■ The pair $\{s\sigma, s\omega\}$ on the Figure 2.2 defines the unique right triangle $(p, h; a)$ of the legs $p = s\omega$ and h and hypotenuse $a = \mathrm{s}\sigma$. It's unique companions are the right triangle $(a, b; w)$ inscribed in the circle of a diameter equal to its hypotenuse w and its complementary right triangle $(q, h; b)$ in the triangle $(a, b; w)$. Consequently $p + q = w$ and $pq = h^2$.

We associate the equation (2) to the right triangle $(p, h; a)$ on the Figure 2.1 by the identifications

$$p = \mathrm{s}\omega, \quad (\eta + \xi)(\eta - \xi) \ = h^2 = \ (\mathrm{s}\sigma)^2 - (\mathrm{s}\omega)^2.$$

Since $pq = h^2 = \mathrm{s}\sigma\, q$ by comparison we obtain the following two systems of the linear equations

$$\begin{aligned} &1. \quad \eta + \xi = \mathrm{s}\omega, \quad q = \eta - \xi \\ &2. \quad \eta - \xi = \mathrm{s}\omega, \quad p = \eta + \xi. \end{aligned}$$

Each linear equation is coupled to the second Euler Box equation $\eta^2 + \xi^2 = 2 - \mathrm{s}^2$ making two systems of the linear-quadratic equation couples. We introduce the variable $\chi \in \{\xi, -\xi\}$ to write both systems in the unified form

$$\eta+\chi=\mathrm{s}\omega,\ \eta^2+\chi^2=2-\mathrm{s}^2$$
$$\Updownarrow$$
$$\eta+\chi=\mathrm{s}\omega,\quad 2\eta\chi=\mathrm{s}^2(1+\omega^2)-2.$$

Hence, the variables (η,χ) are the solutions to the quadratic equation

$$\mathrm{Z}^2-\mathrm{s}\omega\mathrm{Z}+2^{-1}[\mathrm{s}^2(1+\omega^2)-2\,]=0$$

in the Z. Whenever the discriminant $D=2+\mathrm{s}^2$ is the perfect square, the η and ξ asserted above are the rational numbers. □

Conclusion

There is no Perfect Cuboid.

■ The rational number nature of the Pythagorean pair (s,c) and a perfect square integer determinant is required by the Perfect Cuboid. Thus, there must exist a rational number R such that

$$D=2+\mathrm{s}^2=3-\mathrm{c}^2=\mathrm{R}^2\quad\Leftrightarrow\quad \mathrm{c}^2+\mathrm{R}^2=3.$$

If the rational solution exists there must be a minimal integer triple $\{k,l,m\}\colon klm\neq 0,(k,l,m)=1$ such that

$$\mathrm{c}^2+\mathrm{R}^2=m^{-2}(k^2+l^2)=3\quad\Leftrightarrow\quad k^2+l^2=3m^2.$$

Either $3|(k^2+l^2)$ or $3|k^2$ and $3|l^2$. If $3|(k^2+l^2)$ the equality of the reminders implies

$$[(k^2+l^2):3]=0=[3m^2:3]=m^2\quad\therefore\quad m=0\ \Rightarrow\ k=l=0.$$

The Perfect Cuboid is trivial. Else, there are integers (u,v) such that $k=3u$, $l=3v$ so that

$$k^2+l^2=3m^2\ \Leftrightarrow\ 9(u^2+v^2)=3m^2\ \Leftrightarrow\ 3(u^2+v^2)=m^2=(3t)^2\ \Leftrightarrow\ u^2+v^2=3t^2.$$

Thus, there is an integer triple (u,v,t) smaller than the minimal $\{k,l,m\}$, satisfying the same equation which is not possible by the Fermat Descent Principle. In addition to trivia, there are no other integers to make the determinant a perfect square. Hence $r\cdot(\sqrt{\Phi},\sqrt{\Psi})$ are the multiplication rational, which is r is not rational and the Euler Box cannot be the Perfect Cuboid. □

References

[1] W. E. Deskins, *Abstract Algebra*, The MacMilan Company, New York,

[2] George E. Andrews, *Number Theory*, Dower Publications, Inc. New York.