On the Decomposition of the Pochhammer's symbol

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And Jesus said unto them, I am the bread of life: he that cometh to me shall never hunger; and he that believe on me shall never thirst. John 6:35.

ABSTRACT. I derive an identity for the decomposition of the Pochhamer's symbol.

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1. INTRODUCTION

In present paper, I derive the identity below

$$(a)_n = \frac{a}{a+n} \cdot \frac{(a+1)_{k-1}}{(a+n+1)_{2k-1}} \cdot (a+k)_{n+k}.$$

2. PRELIMINARY

Lemma 2.1. *Let u_r and v_r two sequences, then*

$$\prod_{r=1}^{n} \frac{u_r}{v_r} = 1 + \sum_{k=1}^{n} \left(1 - \frac{v_k}{u_k}\right) \prod_{r=1}^{k} \frac{u_r}{v_r}. \tag{2.1}$$

provided none of the denominators in (2.1) are zero.

Proof. See [1, p. 4, (2.2)]. □

3. THE MAIN THEOREM

3.1. The Finite sum for $(a)_n/(b)_n$.

Lemma 3.1. *If $a,b\in\mathbb{R}$ and $n\in\mathbb{Z}^+$, then*

$$\frac{(a)_n}{(b)_n}=\frac{a(n+b)}{b(n+a)}\left[1+\sum_{k=1}^{n}\left(\frac{a-b}{a+k}\right)\frac{(a+1)_k}{(b+1)_k}\right], \tag{3.1}$$

provided none of the denominators in (3.1) are zero.

Proof. Consider the function defined by

$$E_n(a,b):=\frac{(a)_n}{(b)_n} \tag{3.2}$$

I know [2, p. 1, (4)] that

$$(\ell)_n=\prod_{r=0}^{n-1}(\ell+r). \tag{3.3}$$

From (3.2) and (3.3), I conclude that

$$E_n(a,b)=\prod_{r=0}^{n-1}\frac{a+r}{b+r}=\frac{a(n+b)}{b(n+a)}\prod_{r=1}^{n}\frac{a+r}{b+r}. \tag{3.4}$$

On the other hand, replace u_r by $a+r$ and v_r by $b+r$ in Lemma 2.1

$$\begin{aligned}\prod_{r=1}^{n}\frac{a+r}{b+r}&=1+\sum_{k=1}^{n}\left(1-\frac{b+k}{a+k}\right)\prod_{r=1}^{k}\frac{a+r}{b+r}\\&=1+\sum_{k=1}^{n}\frac{a-b}{a+k}\prod_{r=1}^{k}\frac{a+r}{b+r}\\&=1+\sum_{k=1}^{n}\left(\frac{a-b}{a+k}\right)\frac{(a+1)_k}{(b+1)_k}.\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), it follows that

$$E_n(a,b)=\frac{a(n+b)}{b(n+a)}\left[1+\sum_{k=1}^{n}\left(\frac{a-b}{a+k}\right)\frac{(a+1)_k}{(b+1)_k}\right]. \tag{3.6}$$

Now, from (3.2) and (3.6), I conclude that

$$\frac{(a)_n}{(b)_n}=\frac{a(n+b)}{b(n+a)}\left[1+\sum_{k=1}^{n}\left(\frac{a-b}{a+k}\right)\frac{(a+1)_k}{(b+1)_k}\right],$$

which is the desired result. □

3.2. The Decomposition of $(a)_n/(b)_n$.

Theorem 3.2. *If $a,b\in\mathbb{R}$, $n\in\mathbb{Z}^+$ and $k\in\mathbb{Z}^+$, then*

$$\frac{(a)_n}{(b)_n}=\frac{a}{b}\cdot\frac{b+n}{a+n}\cdot\frac{(a+1)_{k-1}}{(b+1)_{k-1}}\cdot\frac{(b+n+1)_{2k-1}}{(a+n+1)_{2k-1}}\cdot\frac{(a+k)_{n+k}}{(b+k)_{n+k}}, \tag{3.7}$$

provided none of the denominators in (3.7) are zero.

Proof. Suppose the definition below

$$S_n(a,b) := \frac{b(n+a)}{a(n+b)} \cdot \frac{(a)_n}{(b)_n} = 1 + \sum_{k=1}^{n} \left(\frac{a-b}{a+k}\right)\frac{(a+1)_k}{(b+1)_k}, \tag{3.8}$$

by virtue of the Lemma 3.1.

Replace n by $n+1$ in the right hand side of (3.8)

$$\begin{aligned} S_{n+1}(a,b) &= 1 + \sum_{k=1}^{n+1} \left(\frac{a-b}{a+k}\right)\frac{(a+1)_k}{(b+1)_k} \\ &= 1 + \sum_{k=1}^{n} \left(\frac{a-b}{a+k}\right)\frac{(a+1)_k}{(b+1)_k} + \left(\frac{a-b}{a+n+1}\right)\frac{(a+1)_{n+1}}{(b+1)_{n+1}} \\ &= S_n(a,b) + \left(\frac{a-b}{a+n+1}\right)\frac{(a+1)_{n+1}}{(b+1)_{n+1}} \\ \Rightarrow S_{n+1}(a,b) - S_n(a,b) &= \left(\frac{a-b}{a+n+1}\right)\frac{(a+1)_{n+1}}{(b+1)_{n+1}}. \end{aligned} \tag{3.9}$$

On the other hand, by definition above, I have

$$S_{n+1}(a,b) - S_n(a,b) = \frac{b(n+a+1)}{a(n+b+1)} \cdot \frac{(a)_{n+1}}{(b)_{n+1}} - \frac{b(n+a)}{a(n+b)} \cdot \frac{(a)_n}{(b)_n}. \tag{3.10}$$

From (3.9) and (3.10), it follows that

$$\frac{b(n+a+1)}{a(n+b+1)} \cdot \frac{(a)_{n+1}}{(b)_{n+1}} - \frac{b(n+a)}{a(n+b)} \cdot \frac{(a)_n}{(b)_n} = \left(\frac{a-b}{a+n+1}\right)\frac{(a+1)_{n+1}}{(b+1)_{n+1}}. \tag{3.11}$$

With a bit of manipulation (3.11) becomes

$$\frac{(a)_n}{(b)_n} = \frac{(n+a+1)(n+b)}{(n+a)(n+b+1)} \cdot \frac{(a)_{n+1}}{(b)_{n+1}} - \frac{a(a-b)(n+b)}{b(n+a)(a+n+1)} \cdot \frac{(a+1)_{n+1}}{(b+1)_{n+1}}. \tag{3.12}$$

Note that

$$\frac{(a+1)_{n+1}}{(a)_{n+1}} = \frac{n+a+1}{a} \Rightarrow (a)_{n+1} = \frac{a(a+1)_{n+1}}{n+a+1} \tag{3.13}$$

and

$$\frac{(b+1)_{n+1}}{(b)_{n+1}} = \frac{n+b+1}{b} \Rightarrow (b)_{n+1} = \frac{b(b+1)_{n+1}}{n+b+1}. \tag{3.14}$$

Divide (3.13) by (3.14)

$$\frac{(a)_{n+1}}{(b)_{n+1}} = \frac{a(n+b+1)}{b(n+a+1)} \cdot \frac{(a+1)_{n+1}}{(b+1)_{n+1}}. \tag{3.15}$$

From (3.12) and (3.15), I conclude that

$$\frac{(a)_n}{(b)_n} = \frac{a(n+b)(n+b+1)}{b(n+a)(n+a+1)} \cdot \frac{(a+1)_{n+1}}{(b+1)_{n+1}}. \tag{3.16}$$

Replace a by $a+1$, b by $b+1$ and n by $n+1$ in (3.16)

$$\frac{(a+1)_{n+1}}{(b+1)_{n+1}}=\frac{(a+1)(n+b+2)(n+b+3)}{(b+1)(n+a+2)(n+a+3)}\cdot\frac{(a+2)_{n+2}}{(b+2)_{n+2}}. \tag{3.17}$$

Substitute the right hand side of (3.17) in the right hand side of (3.16)

$$\frac{(a)_n}{(b)_n}=\frac{a(a+1)(n+b)(n+b+1)(n+b+2)(n+b+3)}{b(b+1)(n+a)(n+a+1)(n+a+2)(n+a+3)}\cdot\frac{(a+2)_{n+2}}{(b+2)_{n+2}}. \tag{3.18}$$

Replace a by $a+2$, b by $b+2$ and n by $n+2$ in (3.16)

$$\frac{(a+2)_{n+2}}{(b+2)_{n+2}}=\frac{(a+2)(n+b+4)(n+b+5)}{(b+2)(n+a+4)(n+a+5)}\cdot\frac{(a+3)_{n+3}}{(b+3)_{n+3}}. \tag{3.19}$$

Substitute the right hand side of (3.19) in the right hand side of (3.18)

$$\begin{aligned}\frac{(a)_n}{(b)_n}&=\frac{a(a+1)(a+2)}{b(b+1)(b+2)}\\ &\cdot\frac{(n+b)(n+b+1)(n+b+2)(n+b+3)(n+b+4)(n+b+5)}{(n+a)(n+a+1)(n+a+2)(n+a+3)(n+a+4)(n+a+5)}\cdot\frac{(a+3)_{n+3}}{(b+3)_{n+3}}.\end{aligned} \tag{3.20}$$

I note easily that for the k-th iteration, I get

$$\frac{(a)_n}{(b)_n}=\left(\prod_{r=0}^{k-1}\frac{a+r}{b+r}\right)\left(\prod_{r=0}^{2k-1}\frac{n+b+r}{n+a+r}\right)\cdot\frac{(a+k)_{n+k}}{(b+k)_{n+k}}. \tag{3.21}$$

On the other hand, I know [2, p. 1, (4)] that

$$\prod_{r=1}^{k}(a+r-1)=(a)_k. \tag{3.22}$$

Applying (3.22) in the right hand side of (3.21), I encounter

$$\frac{(a)_n}{(b)_n}=\frac{a}{b}\cdot\frac{b+n}{a+n}\cdot\frac{(a+1)_{k-1}}{(b+1)_{k-1}}\cdot\frac{(b+n+1)_{2k-1}}{(a+n+1)_{2k-1}}\cdot\frac{(a+k)_{n+k}}{(b+k)_{n+k}},$$

which is the desired result. □

Corollary 3.3. *If* $a\in\mathbb{R}$, $n\in\mathbb{Z}^+$ *and* $k\in\mathbb{Z}^+$, *then*

$$(a)_n=\frac{a}{a+n}\cdot\frac{(a+1)_{k-1}}{(a+n+1)_{2k-1}}\cdot(a+k)_{n+k}, \tag{3.23}$$

provided none of the denominators in (3.23) are zero.

Proof. Separate the variables a and b from the Theorem 3.2 and compare the both members. □

REFERENCES

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4. Appendix

I present below a new proof for an old identity of finite sum:

Corollary 4.1. *If* $|x|<1$, $x\neq 0$ *and* $n\in\mathbb{Z}^+$, *then*

$$\sum_{k=0}^{n-1} x^k = \frac{x^n-1}{x-1}. \tag{4.1}$$

Proof. In [3, p. 4, Corollary 3.2], I have the following limit formula for the n-power

$$x^n = \lim_{\ell\to\infty} \frac{(x\ell)_n}{(\ell)_n} \tag{4.2}$$

Replace a by $x\ell$ and b by ℓ in Lemma 3.1

$$\begin{aligned}\frac{(x\ell)_n}{(\ell)_n} &= \frac{x\ell(n+\ell)}{\ell(n+x\ell)}\left[1+\sum_{k=1}^{n}\left(\frac{x\ell-\ell}{x\ell+k}\right)\frac{(x\ell+1)_k}{(\ell+1)_k}\right] \\ &= x\frac{n+\ell}{n+x\ell}\left[1+\sum_{k=1}^{n}\left(\frac{x\ell-\ell}{x\ell+k}\right)\frac{(x\ell+1)_k}{(\ell+1)_k}\right].\end{aligned} \tag{4.3}$$

From (4.2) and (4.3), I conclude that

$$x^n = x\lim_{\ell\to\infty}\frac{n+\ell}{n+x\ell}\left[1+\sum_{k=1}^{n}\left(\frac{x\ell-\ell}{x\ell+k}\right)\frac{(x\ell+1)_k}{(\ell+1)_k}\right]. \tag{4.4}$$

On the other hand, I know that

$$\lim_{\ell\to\infty}\frac{n+\ell}{n+x\ell} = \frac{1}{x}, \tag{4.5}$$

$$\lim_{\ell\to\infty}\frac{x\ell-\ell}{x\ell+k} = \frac{x-1}{x} = 1-\frac{1}{x} \tag{4.6}$$

and

$$\lim_{\ell\to\infty}\frac{(x\ell+1)_k}{(\ell+1)_k} = x^k, \tag{4.7}$$

From (4.4),(4.5),(4.6) and (4.7), it follows that

$$\begin{aligned} x^n &= 1+\left(1-\frac{1}{x}\right)\sum_{k=1}^{n} x^k \\ \Rightarrow x^n-1 &= \left(1-\frac{1}{x}\right)\sum_{k=1}^{n} x^k \\ \Rightarrow \frac{x^n-1}{x-1} &= \sum_{k=1}^{n} x^{k-1} \\ \Rightarrow \sum_{k=0}^{n-1} x^k &= \frac{x^n-1}{x-1}, \end{aligned} \tag{4.8}$$

which is the desired result. □