The generalized Seiberg-Witten equations

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Abstract

We show a set of equations which generalizes the Seiberg-Witten equations, we show also compacity of the moduli spaces.

1 Recalls of differential geometry

The $Spin-C$-structures are reductions of a $SO(n).S^1$- fiber bundle to the group $Spin^C(n) = Spin(n) \times_{\{1,-1\}} S^1$. For a four-manifold it exists always a $Spin-C$-structure for the tangent fiber bundle [F].

The Dirac operator is defined over the $Spin-C$-structure with help of a connection A for the associated line bundle.

$$\mathcal{D}_A = \sum_i e_i.\nabla^A_{e_i}$$

with ∇^A the connection defined by the Levi-Civita connection and the connection A of the determinant fiber bundle of the $Spin-C$-structure.

The self-dual part of the curvature (which is a 2-form) of the connection A is considered:

$$\Omega^+_A$$

A self-dual 2-form with imaginary values, bound to a spinor $\psi \in S^+$ is also defined by [F]:

$$\omega(\psi)(X,Y) = < X.Y.\psi, \psi > + < X, Y > |\psi|^2$$

2 The Seiberg-Witten equations

The Seiberg-Witten equations are the following ones [F] [M]:
1)

$$\mathcal{D}_A(\psi) = 0$$

2)

$$\Omega^+_A = -(1/4)\omega(\psi)$$

3 The generalization of the SW equations

We consider two spinors ψ, ϕ and we define [F] the coupled Seiberg-Witten equations for $(A, A', f, g, \psi, \phi)$:
1)

$$\mathcal{D}_A(\psi) = 0$$

2)

$$\mathcal{D}_{A'}(\phi) = 0$$

3)

$$\Omega_A^+ = -(1/4)\omega(\psi)$$

4)

$$\Omega_{A'}^+ = -(1/4)\omega(\phi)$$

5)

$$A - A' = \frac{d < \psi|\phi >}{< \psi|\phi >}$$

A, A' are connections $f, g : M \to S^1$.

The gauge group acts:

$$(h, h').(A, A', \psi, \phi) = ((1/h)^* A, (1/h')^* A', h\psi, h'\phi)$$

Moreover, the situation can be generalized to n solutions of the Seiberg-Witten equations:
1)

$$\mathcal{D}_{A_i}(\psi_i) = 0$$

2)

$$\Omega_{A_i}^+ = -(1/4)\omega(\psi_i)$$

3)

$$A_i - A_j = \frac{d < \psi_i|\psi_j >}{< \psi_i|\psi_j >}$$

4 The compacity of the generalized SW moduli spaces

We define:

$$M_L = \{(\psi, \phi, A, A') \in \Gamma(S^+)^2.C(P)^2 : \mathcal{D}_A\psi = \mathcal{D}_{A'}\phi = 0,$$

$$\Omega_A^+ = -(1/4)\omega(\psi), \Omega_{A'}^+ = -(1/4)\omega(\phi), A - A' = \frac{d < \psi|\phi >}{< \psi|\phi >}\}/\mathcal{G}^2(P)$$

Theorem 1 *M_L is compact.*

Proof : It is a closed set in the product of two compact sets. (The proof is given in [F] P136-137.)

References

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