The generalized Seiberg-Witten equations

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Abstract

We show a set of equations which generalizes the Seiberg-Witten equations

1 Recalls of differential geometry

The $Spin-C$-structures are reductions of a $SO(n).S^1$- fiber bundle to the group $Spin(n)\times_{\{1,-1\}}S^1$. For a four-manifold it exists always a $Spin-C$-structure for the tangent fiber bundle [F].

The DIrac operator is define over the $Spin-C$-structure with help of a connection A for the associated line bundle.

$$\mathcal{D}_A=\sum_i e_i.\nabla^A_{e_i}$$

with ∇^A the connection defined by the Levi-Civita connection and the connection A.

The self-dual part of the curvature (which is a 2-form) of the connection A is considered:

$$\Omega^+_A$$

A 2-form bound to a spinor ψ is also defined by [F]:

$$\omega(\psi)(X,Y)=<X.Y.\psi,\psi>+<X,Y>|\psi|^2$$

2 The Seiberg-Witten equations

The Seiberg-Witten equations are the following ones [F] [M]:
1)

$$\mathcal{D}_A(\psi)=0$$

2)

$$\Omega^+_A=-(1/4)\omega(\psi)$$

3 The generalization of the SW equations

We consider two spinors ψ, ϕ and we define [F] the coupled Seiberg-Witten equations (A, A', f, ψ, ϕ):
1)
$$\mathcal{D}_A(f\psi) = 0$$
2)
$$\mathcal{D}_{A'}((1/f)\phi) = 0$$
3)
$$\Omega_A^+ = -(1/4)\omega(\psi)$$
4)
$$\Omega_{A'}^+ = -(1/4)\omega(\phi)$$
5)
$$(f^2)^* A = (1/f^2)^* A'$$

A, A' are connections $f : M \to S^1$. If $f = e$, then we have the Seiberg-Witten equations.

The gauge group acts:

$$g.(A, A', f, \psi, \phi) = ((1/g^2)^* A, (g^2)^* A', fg, g\psi, (1/g)\phi)$$

Moreover, the situation can be generalized to n solutions of the Seiberg-Witten equations:
1)
$$\mathcal{D}_{A_i}(f_i\psi_i) = 0$$
2)
$$\Omega_{A_i}^+ = -(1/4)\omega(\psi_i)$$
3)
$$(f_i^2)^* A_i = B$$
4)
$$\prod_i f_i = 1$$

4 The compacity of the generalized SW moduli spaces

Theorem 1 *Let (ψ, A) be a solutions of $\mathcal{D}_A\psi = 0, \Omega_A^+ = -(1/4)\omega(\psi)$ over a compact Riemann manifold (M, g) with scalar curvature R. Then at each point,*
$$|\psi(x)|^2 \leq -R_{min}$$
with $R_{min} = min\{R(m), m \in M\}$

The proof is given in [F] p135.

Definition 1 *We define:*

$$M_L = \{(\psi, \phi, A, A', f) \in \Gamma(S^+)^2.C(P)^2.Map(M, S^1) : \mathcal{D}_A\psi = \mathcal{D}_{A'}\phi = 0,$$
$$\Omega_A^+ = -(1/4)\omega(\psi), \Omega_{A'}^+ = -(1/4)\omega(\phi), (f^2)^* A = (1/f^2)^* A'\}/G$$

Theorem 2 *M_L is compact.*

Proof : Let

$$F(L) = \{\omega \in \Lambda(M) : d\omega = 0, [\omega]_{DR} = c_1(L)\}$$

Since the curvature form is gauge invariant, we obtain a mapping:

$$P : M_L \to F(P), P[A, A', \psi, \phi, f] = \Omega_A = \Omega_{A'}$$

4.1 First step

$P(M_L) \to F(L)$ is a compact subset.

The proof is given in [F] P136-137.

4.2 Second step

Let be $P_1, P_2 : M_L \to \mathcal{C}(P)/G(P)$,

$$P_1(\psi, \phi, A, A', f) = A,$$

and

$$P_2(\psi, \phi, A, A', f) = A',$$

then $P_1, P_2(M_L) \subset \mathcal{C}(P)/G(P)$ are compact subsets. We use Weyl's theorem. The mapping $\mathcal{C}(P) \to F(P), A \to \Omega_A$ is a fibration with compact fibre $Pic(M) = H^1(M, \mathbf{R})/H^1(M, \mathbf{Z})$. The following diagram commutes:

$$\begin{array}{ccc} M_L & \to^{P_1} & \mathcal{C}(P)/G(P) \\ \downarrow^P & & \downarrow \\ F(L) & = & F(L) \end{array}$$

$P_1, P_2(M) \subset \mathcal{C}(P)/G(P)$ are compact subsets.

4.3 Third step

Let be $F : M_L \to G(P)$, $F(\psi, \phi, A, A', f) = f$, then $F(M_L) \subset G(P)$ is a compact subset.

Consider the map: $K : M_L \to \Lambda^1(M)$,

$$K(A, A', \psi, \phi, f) = \frac{df}{f}$$

then $K(M_L) \subset \Lambda^1(M)$ is compact. Indeed, $4\frac{df}{f} = A' - A$ which is compact. And the fiber is $\frac{df}{f} = \frac{df'}{f'}$, $f/f' = cst \in S^1$

4.4 Fourth step: M_L is compact

$P_1^{-1}(A), P_2^{-1}(A'), F^{-1}(\alpha)$ consists of the solutions of

$$\mathcal{D}_A f\psi = \mathcal{D}_{A'}(1/f)\phi = 0, max(|\psi(x)|, |\phi(x)|) \leq -R_{min}$$

This is bounded ball in a finite-dimensional vector space.

References

[B] N.Berline, E.Getzler, M.Vergne, ”Heat kernels and Dirac operators”, Springer-Verlag, 1992.

[F] T.Friedrich, ”Dirac operators in Riemannian Geometry”, Graduate Studies in Mathematics vol 25, AMS, 2000.

[K] M.Karoubi, ”Algèbres de Clifford et K-théorie”, Ann.Scient.Ec.Norm.Sup. 4 ser. 1 (1968), 161-270.

[M] J.Morgan, ”The Seiberg-Witten equations and applications to the topology of smooth four-manifolds”, Mathematical Notes, Princeton University Press, 1996.

[W] E.Witten, ”Monopoles and four-manifolds”, Math.Res.Lett. 1(1994), 769-796.