Lyapunov-type inequality for the Hadamard fractional boundary value problem on a general interval $[a;b], (1 \leqslant a < b)$

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Abstract: In this paper, we study an open problem; where we obtained several results for a Lyapunov-type and Hartman-Wintner-type inequalities for a Hadamard fractional differential equation on a general interval $[a;b], (1 \leqslant a < b)$ with the boundary value conditions.

Keywords: Hadamard fractional derivative; boundary value problem; Green's function; Lyapunov-type inequality; Hartman-Wintner-type inequality.

1 introduction

The first result in this domain is due to Lyaponov [1], can be stated as follows:
    If a nontrivial continuous solution to the following boundary value problem

$$
\begin{cases}
u''(t) + q(t)u(t) = 0, & a < t < b, \\
u(a) = u(b) = 0,
\end{cases}
\tag{1}
$$

existe, where $q : [a;b] \to \mathbb{R}$ is a continuous function, then

$$
\int_a^b |q(s)|ds > \frac{4}{b-a}.
\tag{2}
$$

Recently, several articles from the inequality of Lyapunov have been published about a differential equations of the integer order and fractional order, see [5-10] and references therein, for example:
The following result for the Riemann-Liouville fractional boundary value problem is found by D. O'Regan and B. Samet [4]

$$
\begin{cases}
{}^{R}D^\alpha u(t) + q(t)u(t) = 0, & a < t < b, \quad 3 < \alpha \leqslant 4, \\
u(a) = u'(a) = u''(a) = u''(b) = 0,
\end{cases}
\tag{3}
$$

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has a nontrivial continuous solution, then

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)(\alpha-2)^{\alpha-2}}{2(\alpha-3)^{\alpha-3}(b-a)^{\alpha-1}}.$$ (4)

In [2] Qinghua, Chao and Jinxun established a Lyapunov-type inequality for a differential equation that depands on the Hadamard fractional derivative, for the boundary value problem

$$\begin{cases} {}^HD^\alpha u(t) - q(t)u(t) = 0, & 1 < t < e, \quad 1 < \alpha \leqslant 2, \\ u(1) = u(e) = 0, \end{cases}$$ (5)

where $q : [1; e] \to \mathbb{R}$ is a continuous function.They proved that if a nontrivial continuous solution to the above problem, then

$$\int_1^e |q(s)|ds > \Gamma(\alpha)\lambda^{1-\alpha}(1-\lambda)^{1-\alpha}\exp\lambda,$$ (6)

where $\lambda = \frac{2\alpha-1-\sqrt{(2\alpha-2)^2+1}}{2}$.

and they have presented the following open problem for readers:

How to get the Lyapunov inequality for the following the Hadarmard fractional value problem (HFBVP)

$$\begin{cases} {}^HD^\alpha u(t) - q(t)u(t) = 0, & 1 \leqslant a < t < b, \quad 1 < \alpha \leqslant 2, \\ u(a) = u(b) = 0, \end{cases}$$ (7)

where ${}^HD^\alpha$ is the Hadamard fractional derivative, and $q : [a; b] \to \mathbb{R}$ is a continuous function.

In this paper we answered the previous question by using two methods, and also we get the Hartman-Wintner-type inequalities.

2   Preliminaries

Definition 1 *[3] Let $a, b, \alpha \in \mathbb{R}^+$ where $a < b$ and $n - 1 < \alpha \leqslant n$ with $n \in \mathbb{N}^*$, The Hadamard fractional integral of ordre α.for a function $f \in L^1[a, b]$ is defined as*

$${}^H_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}, \ a < t < b$$ (8)

with Γ is Gamma Euler function

Definition 2 *[3] Let $a, b \in \mathbb{R}^+$ with $a < b$, The Hadamard fractional derivative of ordre $\alpha \in \mathbb{R}^+$.for a function $f \in L^1[a, b]$ is defined as*

$${}^H_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}t^n\frac{d^n}{dt^n}\int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} f(s)\frac{ds}{s}, \ a < t < b$$ (9)

where $n - 1 < \alpha \leqslant n$ with $n \in \mathbb{N}^*$

Lemma 3 *[3] Let $0 \leqslant a < b$ and $\alpha > 0$ where $n - 1 < \alpha \leqslant n$. and $n \in \mathbb{N}^*$ The equation $^H D^\alpha u(t) = 0$ has as its solutions*

$$u(t) = \sum_{j=1}^{j=n} c_i \left(\ln \frac{t}{a} \right)^{\alpha - i}, \quad t \in [a, b] \tag{10}$$

and moreover

$$^H I^\alpha \ ^H D^\alpha u(t) = u(t) + \sum_{j=1}^{j=n} c_i \left(\ln \frac{t}{a} \right)^{\alpha - i}, \tag{11}$$

where $c_i \in \mathbb{R}$, $(i = 1, ..., n)$ are constants.

Lemma 4 *Let $A, B \in \mathbb{R}$, we have*

$$AB \leqslant \frac{(A + B)^2}{4} \tag{12}$$

3 Main results

Lemma 5 *Let $u \in C([a; b], \mathbb{R})$, the following problem*

$$\begin{cases} ^H D^\alpha u(t) - q(t) u(t) = 0, & 1 \leqslant a < t < b, \quad 1 < \alpha \leqslant 2, \\ u(a) = u(b) = 0, \end{cases} \tag{13}$$

has equivalent to the fractional integral equation

$$u(t) = \int_0^t G(t, s) q(s) u(s) \, ds \tag{14}$$

where

$$G(t, s) = \begin{cases} g_1(t, s) = g_2(t, s) + \frac{1}{\Gamma(\alpha)} \left(\ln \frac{t}{s} \right)^{\alpha - 1} \frac{1}{s}, & a \leqslant s \leqslant t \leqslant b, \\ g_2(t, s) = -\frac{1}{\Gamma(\alpha)} \frac{\left(\ln \frac{t}{a} \right)^{\alpha - 1} \left(\ln \frac{b}{s} \right)^{\alpha - 1}}{\left(\ln \frac{b}{a} \right)^{\alpha - 1}} \frac{1}{s}, & a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{15}$$

with $1 \leqslant a < b$.

Proof. Using Lemma 3, we have

$$u(t) = c_1 \left(\ln \frac{t}{a} \right)^{\alpha - 1} + c_2 \left(\ln \frac{t}{a} \right)^{\alpha - 2} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s} \right)^{\alpha - 1} q(s) u(s) \frac{ds}{s} \tag{16}$$

where $c_1, c_2 \in \mathbb{R}$
using the boundary condition $u(a) = u(b) = 0$ we get $c_2 = 0$ and

$$c_1 = -\frac{\left(\ln \frac{b}{a} \right)^{1 - \alpha}}{\Gamma(\alpha)} \int_a^b \left(\ln \frac{b}{s} \right)^{\alpha - 1} q(s) u(s) \frac{ds}{s} \tag{17}$$

Substituting the values of c_1 and c_2 in (16), we obtain

$$
\begin{aligned}
u\left(t\right) &= -\frac{\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{a}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-1}q(s)u\left(s\right)\frac{ds}{s} \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}q(s)u\left(s\right)\frac{ds}{s} \\
&= \frac{1}{\Gamma(\alpha)}\int_a^t\left[\left(\ln\frac{t}{s}\right)^{\alpha-1}-\left(\ln\frac{b}{a}\right)^{1-\alpha}\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1}\right]q(s)u\left(s\right)\frac{ds}{s} \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_t^b\left(\ln\frac{b}{a}\right)^{1-\alpha}\left(\ln\frac{t}{a}\right)^{\alpha-1}\left(\ln\frac{b}{s}\right)^{\alpha-1}q(s)u\left(s\right)\frac{ds}{s} \\
&= \int_a^t G\left(t,s\right)q(s)u\left(s\right)ds \qquad\qquad\qquad (18)
\end{aligned}
$$

the proof is complete ∎

Lemma 6 *The Green's function G defined in Lemma 5, has the following properties*
i) $G(t,s)\leqslant g_2(s,s)\leqslant 0$, for all $(t,s)\in[a,b]\times[a,b]$
ii) For any $s\in[a,b]$

$$
|G(t,s)|\leqslant|G(s,s)|=-g_2(s,s)\leqslant\frac{1}{4^{(\alpha-1)}\Gamma(\alpha)a}\left(\ln\frac{b}{a}\right)^{\alpha-1} \qquad (19)
$$

Proof. *We start by fixing an arbitrary $s\in[a,b]$. Differentiating $G(t,s)$ with respect to t, we get*
For $1\leqslant a\leqslant t\leqslant s\leqslant b$, we have

$$
\frac{\partial}{\partial t}g_2=-\frac{(\alpha-1)}{\Gamma(\alpha)st}\frac{\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}\leqslant 0, \qquad (20)
$$

we obtain

$$
g_2(s,s)\leqslant g_2(t,s)\leqslant g_2(a,s)=0, \qquad (21)
$$

while for $1\leqslant a\leqslant s\leqslant t\leqslant b$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}g_1 &= \frac{\partial}{\partial t}g_2+\frac{(\alpha-1)}{\Gamma(\alpha)st}\left(\ln\frac{t}{s}\right)^{\alpha-2} \\
&= -\frac{(\alpha-1)}{\Gamma(\alpha)st}\frac{\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}+\frac{(\alpha-1)}{\Gamma(\alpha)st}\left(\ln\frac{t}{s}\right)^{\alpha-2} \\
&= \frac{(\alpha-1)}{\Gamma(\alpha)st}\left[\left(\ln\frac{t}{s}\right)^{\alpha-2}-\frac{\left(\ln\frac{t}{a}\right)^{\alpha-2}\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}\right]
\end{aligned}
$$

4

$$= \frac{(\alpha-1)\left(\ln\frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha)st}\left[\frac{\left(\ln\frac{t}{s}\right)^{\alpha-2}}{\left(\ln\frac{t}{a}\right)^{\alpha-2}} - \frac{\left(\ln\frac{b}{s}\right)^{\alpha-1}}{\left(\ln\frac{b}{a}\right)^{\alpha-1}}\right]$$

$$= \frac{(\alpha-1)\left(\ln\frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha)st}\left[\left(\frac{\ln\frac{t}{a}}{\ln\frac{t}{s}}\right)^{2-\alpha} - \left(\frac{\ln\frac{b}{s}}{\ln\frac{b}{a}}\right)^{\alpha-1}\right] \tag{22}$$

by $1 \leqslant a \leqslant s \leqslant t \leqslant b$ we get

$$\left(\frac{\ln\frac{t}{a}}{\ln\frac{t}{s}}\right)^{2-\alpha} \geqslant 1, \tag{23}$$

and

$$-\left(\frac{\ln\frac{b}{s}}{\ln\frac{b}{a}}\right)^{\alpha-1} \geqslant -1 \tag{24}$$

using (23) and (24) we obtain

$$\left[\left(\frac{\ln\frac{t}{s}}{\ln\frac{t}{a}}\right)^{\alpha-2} - \left(\frac{\ln\frac{b}{s}}{\ln\frac{b}{a}}\right)^{\alpha-1}\right] \geqslant 0 \tag{25}$$

So thus

$$\frac{\partial}{\partial t}g_1 \geqslant 0 \tag{26}$$

Using $1 \leqslant a \leqslant s \leqslant t \leqslant b$ we get

$$g_1(s,s) \leqslant g_1(t,s) \leqslant g_1(b,s) = 0 \tag{27}$$

We obtain

$$g_2(s,s) \leqslant g_2(t,s) \leqslant g_1(t,s) \leqslant 0, \tag{28}$$

hence

$$G(t,s) \leqslant 0 \tag{29}$$

We prove of

$$|G(s,s)| \leqslant \frac{1}{4^{\alpha-1}\Gamma(\alpha)a}\left(\ln\frac{b}{a}\right)^{\alpha-1} \tag{30}$$

we have $G(s,s) = g_2(s,s) = g_1(s,s) \leqslant g_2(t,s) \leqslant g_1(t,s) \leqslant 0$.
Using Lemma 4, we have

$$\begin{aligned}
|G(s,s)| &= \frac{1}{\Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}s}\left[\left(\ln\frac{s}{a}\right)\left(\ln\frac{b}{s}\right)\right]^{\alpha-1} \\
&\leqslant \frac{1}{4^{\alpha-1}\Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}s}\left[\left(\ln\frac{s}{a} + \ln\frac{b}{s}\right)^2\right]^{\alpha-1} \\
&= \frac{1}{4^{\alpha-1}\Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}s}\left[\left(\ln\frac{b}{a}\right)^2\right]^{\alpha-1}
\end{aligned}$$

5

$$= \frac{1}{4^{\alpha-1}\Gamma(\alpha)s} \left(\ln \frac{b}{a} \right)^{\alpha-1}$$

$$\leqslant \frac{1}{4^{\alpha-1}\Gamma(\alpha)a} \left(\ln \frac{b}{a} \right)^{\alpha-1}$$

Therefore

$$|G(t,s)| \leqslant |G(s,s)| = -g_2(s,s) \leqslant \frac{1}{4^{(\alpha-1)}\Gamma(\alpha)a} \left(\ln \frac{b}{a} \right)^{\alpha-1} \qquad (31)$$

The proof is complete ∎

We have the following Hartman-Wintner-type inequality.

Theorem 7 *If a nontrivial continuous solution to the Hadamard fractional boundary value problem* (7) *existe, then*

$$\int_a^t \frac{1}{s} \left(\ln \frac{s}{a} \ln \frac{b}{s} \right)^{\alpha-1} |q(s)|\, ds \geqslant \left(\ln \frac{b}{a} \right)^{\alpha-1} \Gamma(\alpha) \qquad (32)$$

Proof. Let $E = C\left([a,b],\mathbb{R}\right)$ be the Banach space endowed with the norm

$$\|u\| = \sup_{t \in [a,b]} |u(t)|$$

we have

$$|u(t)| \leqslant \int_a^t |G(t,s)|\,|q(s)|\,|u(s)|\, ds$$

which yields

$$\|u\| \leqslant \|u\| \int_a^t |g_2(s,s)|\,|q(s)|\,|u(s)|\, ds$$

Since u is non trivial, then $\|u\| \neq 0$, so

$$1 \leqslant \int_a^t \frac{1}{\left(\ln \frac{b}{a} \right)^{\alpha-1} \Gamma(\alpha)s} \left(\ln \frac{s}{a} \ln \frac{b}{s} \right)^{\alpha-1} |q(s)|\, ds$$

from which the inequality in (32) follows ∎

Corollary 8 *If a nontrivial continuous solution to the Hadamard fractional boundary value problem existe, then*

$$\int_a^t \left(\ln \frac{s}{a} \ln \frac{b}{s} \right)^{\alpha-1} |q(s)|\, ds \geqslant a \left(\ln \frac{b}{a} \right)^{\alpha-1} \Gamma(\alpha) \qquad (33)$$

Proof. from theorem 7, we have

$$\int_a^t \frac{1}{s} \left(\ln \frac{s}{a} \ln \frac{b}{s} \right)^{\alpha-1} |q(s)|\, ds \geqslant \left(\ln \frac{b}{a} \right)^{\alpha-1} \Gamma(\alpha)$$

nexte we not $\frac{1}{a} \geqslant \frac{1}{s}$
thus we get

$$\int_a^t \left(\ln \frac{s}{a} \ln \frac{b}{s} \right)^{\alpha-1} |q(s)|\, ds \geqslant a \left(\ln \frac{b}{a} \right)^{\alpha-1} \Gamma(\alpha) \tag{34}$$

∎

We have the following Lyapunov-type inequality.

Theorem 9 *If a nontrivial continuous solution to the Hadamard fractional boundary value problem* (7) *existe, then*

$$\int_a^t |q(s)|\, ds \geqslant 4^{(\alpha-1)} \Gamma(\alpha) a \left(\ln \frac{b}{a} \right)^{1-\alpha} \tag{35}$$

We prove this theorem in two methods
Proof. from the corollary 8, we have

$$\int_a^t |q(s)|\, ds \geqslant a \left(\ln \frac{b}{a} \right)^{\alpha-1} \frac{\Gamma(\alpha)}{\max_{s \in [a,b]} h(s)} \tag{36}$$

where

$$h(s) = \left(\ln \frac{s}{a} \ln \frac{b}{s} \right)^{\alpha-1} \tag{37}$$

If $s = a$ or $s = b$ then $h(s) = 0$
Else if $s \in]a, b[$ we differentiate $h(s)$

$$
\begin{aligned}
h'(s) &= \frac{(\alpha-1)}{s \left(\ln \frac{s}{a} \ln \frac{b}{s} \right)^{2-\alpha}} \left(\ln \frac{b}{s} - \ln \frac{s}{a} \right) \\
&= \frac{(\alpha-1) \left(\ln \frac{ab}{s^2} \right)}{s \left(\ln \frac{s}{a} \ln \frac{b}{s} \right)^{2-\alpha}}
\end{aligned}
$$

we have only one solution

$$s_0 = \sqrt{ab} \tag{38}$$

of the equation $h'(s) = 0$ on $]a; b[$. We obtain

$$\max_{s \in [a,b]} h(s) = h(s_0) = \left(\ln \frac{\sqrt{ab}}{a} \ln \frac{b}{\sqrt{ab}} \right)^{\alpha-1} \tag{39}$$

7

We have

$$ab = \sqrt{ab}\sqrt{ab} \Leftrightarrow \ln\frac{\sqrt{ab}}{a} = \ln\frac{b}{\sqrt{ab}} \Leftrightarrow \left(\ln\frac{\sqrt{ab}}{a} - \ln\frac{b}{\sqrt{ab}}\right)^2 = 0$$

$$\Leftrightarrow \quad 4\left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right) = \left(\ln\frac{\sqrt{ab}}{a}\right)^2 + \left(\ln\frac{b}{\sqrt{ab}}\right)^2 + 2\left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)$$

$$\Leftrightarrow \quad \left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right) = \frac{1}{4}\left(\ln\frac{\sqrt{ab}}{a} + \ln\frac{b}{\sqrt{ab}}\right)^2 = \frac{1}{4}\left(\ln\frac{b}{a}\right)^2$$

$$\Leftrightarrow \quad \left(\ln\frac{\sqrt{ab}}{a}\ln\frac{b}{\sqrt{ab}}\right)^{\alpha-1} = \frac{1}{4^{(\alpha-1)}}\left(\ln\frac{b}{a}\right)^{2(\alpha-1)} \tag{40}$$

by (39) and (40)

$$\max_{s\in[a,b]} h(s) = h(s_0) = \frac{1}{4^{(\alpha-1)}}\left(\ln\frac{b}{a}\right)^{2(\alpha-1)} \tag{41}$$

we substiting (41) into (36) we obtain

$$\int_a^t |q(s)|\,ds \geqslant 4^{(\alpha-1)}\Gamma(\alpha)a\left(\ln\frac{b}{a}\right)^{1-\alpha}$$

The proof is complete ∎

We define the constants:

$$\xi_1 = \exp\left(\frac{1}{2}\left[2(\alpha-1) + \ln ba - \sqrt{4(\alpha-1)^2 + \ln^2\frac{b}{a}}\right]\right), \tag{42}$$

and

$$\xi_2 = \exp\left(\frac{1}{2}\left[2(\alpha-1) + \ln ba + \sqrt{4(\alpha-1)^2 + \ln^2\frac{b}{a}}\right]\right). \tag{43}$$

Lemma 10 *The function G defined in Lemma 5, satisfie the following property*

$$\max_{t,s\in[a,b]} |G(t,s)| = \frac{1}{\Gamma(\alpha)\xi_1}\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{\alpha-1}, \tag{44}$$

Proof. we have $\max\limits_{t,s\in[a,b]} |G(t,s)| = \max\limits_{s\in[a,b]} |g_2(s,s)|$

where

$$g_2(s,s) = -\frac{1}{\Gamma(\alpha)\left(\ln\frac{b}{a}\right)^{\alpha-1}}\frac{\left(\ln\frac{s}{a}\ln\frac{b}{s}\right)^{\alpha-1}}{s}$$

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It follows that we only need to get the maximum value of the function

$$f(s) = \frac{\left(\ln \frac{s}{a} \ln \frac{b}{s}\right)^{\alpha-1}}{s} \tag{45}$$

we observe that $f(a) = f(b) = 0$.
If $s \in]a, b[$, differentiate $f(s)$

$$f'(s) = \left[(\alpha - 1) \frac{\ln \frac{b}{s} - \ln \frac{s}{a}}{\left(\ln \frac{s}{a} \ln \frac{b}{s}\right)} - 1\right] \left(\ln \frac{s}{a} \ln \frac{b}{s}\right)^{\alpha-1} \frac{1}{s^2}$$

we have

$$f'(s) = 0 \quad \Leftrightarrow (\alpha - 1)\left(\ln \frac{b}{s} - \ln \frac{s}{a}\right) = \ln \frac{s}{a} \ln \frac{b}{s}$$
$$\Leftrightarrow [2(\alpha - 1) + \ln b + \ln a] \ln s - [(\alpha - 1) + \ln b] \ln a - \ln^2 s - (\alpha - 1) \ln b = 0$$
$$\Leftrightarrow \ln^2 s - [2(\alpha - 1) + \ln ba] \ln s + [(\alpha - 1) \ln ba + \ln b \ln a] = 0$$
$$\Leftrightarrow x^2 - [2(\alpha - 1) + \ln ba] x + [(\alpha - 1) \ln ba + \ln b \ln a] = 0$$

where $x = \ln s$.
we get

$$\begin{cases} x_1 = \frac{[2(\alpha-1)+\ln ba]-\sqrt{\Delta}}{2} = \ln \xi_1 \\ \\ x_2 = \frac{[2(\alpha-1)+\ln ba]+\sqrt{\Delta}}{2} = \ln \xi_2 \end{cases} \tag{46}$$

where

$$\Delta = 4(\alpha - 1)^2 + \ln^2 \frac{b}{a} \tag{47}$$

we have

$$x_2 \quad > \quad \frac{\ln ba + \sqrt{\left(\ln \frac{b}{a}\right)^2}}{2} = \ln b$$

$$\Rightarrow \quad \xi_2 \notin]a; b[$$

Also we have

$$x_1 \quad = \quad \frac{1}{2}\left(2(\alpha - 1) + \ln ba - \sqrt{\left(2(\alpha - 1) + \ln \frac{b}{a}\right)^2 - 4(\alpha - 1)\left(\ln \frac{b}{a}\right)}\right)$$

$$> \quad \frac{1}{2}\left(2(\alpha - 1) + \ln ba - \sqrt{\left(2(\alpha - 1) + \ln \frac{b}{a}\right)^2}\right)$$

$$= \quad \frac{1}{2}\left(2(\alpha - 1) + \ln ba - 2(\alpha - 1) - \ln \frac{b}{a}\right) = \ln a$$

$$\Rightarrow \quad \xi_1 > a$$

and

$$x_1 = \frac{1}{2}\left(2(\alpha-1)+\ln ba - \sqrt{\left(2(\alpha-1)-\ln\frac{b}{a}\right)^2 + 4(\alpha-1)\ln\frac{b}{a}}\right)$$

$$< \frac{1}{2}\left(2(\alpha-1)+\ln ba - \sqrt{\left(2(\alpha-1)-\ln\frac{b}{a}\right)^2}\right)$$

$$= \frac{1}{2}\left(2(\alpha-1)+\ln ba - \left|2(\alpha-1)-\ln\frac{b}{a}\right|\right)$$

$$\leqslant \frac{1}{2}\left(2(\alpha-1)+\ln ba - \left(|2(\alpha-1)| - \left|\ln\frac{b}{a}\right|\right)\right)$$

$$= \frac{1}{2}\left(2(\alpha-1)+\ln ba - \left(2(\alpha-1)-\ln\frac{b}{a}\right)\right) = \ln b$$

$$\Rightarrow \quad \xi_1 < b$$

we obtient $\xi_1 \in]a;b[$

Hence

$$\max_{s\in[a,b]}|f(s)| = \frac{1}{\xi_1}\left(\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}\right)^{\alpha-1} \tag{48}$$

Therefore

$$\max_{t,s\in[a,b]}|G(t,s)| = \frac{1}{\Gamma(\alpha)\xi_1}\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{\alpha-1}, \tag{49}$$

The proof is complete ∎
We have the following Lyapunov-type inequality.

Theorem 11 *If a nontrivial continuous solution to the HFBVP* (7) *existe, then*

$$\int_a^t |q(s)|\,ds \geqslant \Gamma(\alpha)\xi_1\left(\frac{\ln\frac{\xi_1}{a}\ln\frac{b}{\xi_1}}{\ln\frac{b}{a}}\right)^{1-\alpha}, \tag{50}$$

where

$$\xi_1 = \exp\left(\frac{1}{2}\left[2(\alpha-1)+\ln ba] - \sqrt{4(\alpha-1)^2 + \ln^2\frac{b}{a}}\right]\right).$$

Proof. By Lemma 5, the solution of the HFBVP can be written as

$$u(t) = \int_a^t G(t,s)\,q(s)u(s)\,ds$$

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Thus for all $t \in [a, b]$ we have

$$
\begin{aligned}
|u(t)| &\leqslant \int_a^t |G(t,s)|\,|q(s)|\,|u(s)|\,ds \\
&\leqslant \|u\| \int_a^t |G(t,s)|\,|q(s)|\,ds
\end{aligned}
$$

which yields

$$
\|u\| \leqslant \|u\| \int_a^t |G(t,s)|\,|q(s)|\,ds
$$

Since u is non trivial, then $\|u\| \neq 0$, so

$$
1 \leqslant \int_a^t |G(t,s)|\,|q(s)|\,ds
$$

New, an application of Lemma 10, we obtain

$$
\int_a^t |q(s)|\,ds \geqslant \Gamma(\alpha)\xi_1 \left(\frac{\ln \frac{\xi_1}{a} \ln \frac{b}{\xi_1}}{\ln \frac{b}{a}} \right)^{1-\alpha}
$$

The proof is complete ∎

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